Automorphism groups of omega-categorical structures

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The candidate confirms that the work submitted is her own and that appropriate credit has been given where reference has been made to the work of others.

In memory of Piero,

and of his courage.

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Abstract

In this thesis we investigate a method, developed by M. Rubin, for reconstructing ω-categorical structures from their automorphism group. Reconstruction results give conditions under which the automorphism group of a structure $\mathcal{M}$ determines $\mathcal{M}$ up to bi-interpretability or up to bi-definability. In [25], Rubin isolates one such condition, which is related to the definability of point stabilisers in the automorphism group. If the condition holds, the structure is said to have a *weak* $\forall\exists$ *interpretation* and can be recovered from its automorphism group up to bi-interpretability and, in certain cases, up to bi-definability.

We start by describing Rubin's method, and then we clarify the connection between weak $\forall\exists$ interpretations, the definability of point stabilisers and the small index property (another, better known, reconstruction condition). We also give methods for obtaining new weak $\forall\exists$ interpretations from existing ones.

We then examine a large class of combinatorial structures which contains K_n-free graphs, k-hypergraphs and Henson digraphs, and for which the small index property holds. Using a Baire category approach we show how to obtain weak $\forall\exists$ interpretations for all the structures in this class. The method rests on a series of extension lemmas for finite partial isomorphisms, based on work of B. Herwig [16].

Let V be a vector space of dimension $\aleph_0$ over a finite field F: V is ω-categorical, and so are the projective space $\mathrm{PG}(V)$ and the projective symplectic, unitary and orthogonal spaces on V. We find weak $\forall\exists$ interpretations for all the structures whose domain is $\mathrm{PG}(V)$ and whose automorphism group lies between the projective general linear group and the group of projective semilinear transformations. We produce similar results in the projective symplectic, unitary and orthogonal cases. We also give a reconstruction result for the affine group $\mathrm{AGL}(V)$ acting on V by proving that V as an affine space is interpretable in $\mathrm{AGL}(V)$.

Contents

Introduction

The subject of this thesis is the reconstruction of ω-categorical structures from their automorphism group. The question which motivates reconstruction results is the following: if we are given the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a first order structure $\mathcal{M}$, how much do we know about $\mathcal{M}$? This question is only sensible in highly symmetric contexts, such as the class of ω-categorical structures: the Ryll-Nardzewski theorem ensures that these structures have a very rich automorphism group.

The answer depends on:

- the extent to which we know $\mathrm{Aut}(\mathcal{M})$, i.e. whether we know it as an abstract group, as a topological group or as a permutation group with its action on $\mathcal{M}$;
- the extent to which we want to know $\mathcal{M}$, i.e. up to bi–interpretability, or $\mathcal{M}$ with the apparatus of its 0-definable sets.

In general, knowing $\mathrm{Aut}(\mathcal{M})$ for a structure $\mathcal{M}$ as a pure group does not determine $\mathcal{M}$ up to bi-definability, nor up to bi-interpretability. For example, if Ω is a countable set, then $\mathrm{Aut}(\Omega) = \mathrm{Sym}(\Omega)$ if we regard Ω as a pure set (no structure). The line graph of $\langle \Omega, R \rangle$ (see 1.1.5 below) is a structure whose automorphism group is $\mathrm{Aut}(\Omega)$, but which is not bi-definable with Ω. In [13], an example is given of two ω-categorical structures whose automorphism groups are isomorphic as pure groups, and which are not bi-interpretable.

Recall that $\mathrm{Aut}(\mathcal{M})$ has a natural action on $\mathcal{M}$, which extends to $\mathcal{M}^n$ for any

$n \in \mathbb{N}$ by

$$\bar{a}^g = (a_1^g, \dots, a_n^g)$$

for any $\bar{a} \in \mathcal{M}^n$, $g \in \mathrm{Aut}(\mathcal{M})$. This action gives full information about $\mathcal{M}$ in the following sense: two ω-categorical structures have automorphism groups that are isomorphic as permutation groups if and only if the structures are bi-definable, i.e. they have the same 0-definable sets. This happens because in an ω-categorical structure the 0-definable sets are finite unions of orbits in the action of the automorphism group on the structure.

An intermediate step is provided by $\mathrm{Aut}(\mathcal{M})$ as a topological group. Given any subgroup $G \leq \mathrm{Sym}(\Omega)$, where Ω is any countable set, we can endow it with a topology in the following way: let $\bar{a} \in \Omega^n$, then

$$G_{\bar{a}} = \{g \in G \text{ s.t. } \bar{a}^g = \bar{a}\}$$

is the *stabiliser* of $\bar{a}$. Then a basis of open sets for the topology is given by the set of stabilisers of finite tuples of Ω and their cosets. This topology makes G into a topological group. The topology is highly relevant to reconstruction, because two ω-categorical structures whose automorphism groups are isomorphic as topological groups are bi-interpretable (see [1]).

Reconstruction techniques for ω-categorical structures generally seek conditions on $\mathrm{Aut}(\mathcal{M})$ so that the pure group determines the topology or the action on $\mathcal{M}$. One such condition is the *small index property*. It is easily seen that an open subgroup of $\mathrm{Aut}(\mathcal{M})$ has countable index (i.e. it has countably many cosets in $\mathrm{Aut}(\mathcal{M})$). We say that $\mathcal{M}$ has the small index property if the converse is also true, that is, if any subgroup of $\mathrm{Aut}(\mathcal{M})$ having countable index is open, so that the topology is known from the pure group structure. In [22], D. Lascar has shown that if two ω-categorical structures have automorphism groups which are isomorphic as pure groups, and one of them has the small index property, the two structures are bi-interpretable. There is a rich literature concerning small index: the property has been proved for a pure set, the ordered rationals [28], the random graph [17], a countably infinite dimensional vector space over a finite field, also equipped with bilinear forms [10], [11], and more generally all ω-stable ω-categorical structures [17], certain classes of relational structures studied by Herwig, which include k-hypergraphs, K_m-free graphs and Henson digraphs. In a series of papers, Lascar has obtained small index results outside the ω-categorical context: the uncountable case has essentially been solved in a joint paper with Shelah, in which the small index property is proved for uncountable saturated structures of regular cardinality [23]. The property, though, is not known to hold for very familiar ω-categorical examples like the countable homogeneous universal tournament or the countable homogeneous universal partial order.

The latter examples are handled by the reconstruction method used in this thesis. This has been developed by Matatyahu Rubin in [25], and spells out a certain way in which $\mathcal{M}$ can be parameter-interpretable in $\mathrm{Aut}(\mathcal{M})$. As shown by Rubin, under the assumption "no algebraicity", this allows one to recover $\mathcal{M}$ up to bi-definability. Rubin's condition is related to the definability of point stabilisers in $\mathrm{Aut}(\mathcal{M})$. For a transitive structure $\mathcal{M}$, it consists in interpreting $\mathcal{M}$ in $\mathrm{Aut}(\mathcal{M})$ as a conjugacy class $C \subseteq \mathrm{Aut}(\mathcal{M})$ quotiented by an equivalence relation E, definable in the language of groups and with some extra properties, so that $\mathrm{Aut}(\mathcal{M})$ acts on C/E in the same way as it acts on $\mathcal{M}$. When C and E can be found, we say that $\mathcal{M}$ has a weak $\forall\exists$ interpretation.

In his paper [25], Rubin proves the general reconstruction result based on weak $\forall\exists$ interpretations, and he gives a wealth of applications. The examples he treats are all structures in a relational language, where the relations are binary, e.g. the random graph, and, as said above, the countable homogeneous universal poset and the countable homogeneous universal tournament, for which the small index property seems very difficult to prove. Rubin's paper does not seem to have been much investigated: beyond an unpublished paper by A. Singerman [26], nothing has appeared on weak $\forall\exists$ interpretations so far. Singerman finds a weak $\forall\exists$ interpretation for a well-known class of relational structures which do not have the small index property, thus showing that the two reconstruction methods are not equivalent.

The first chapter of the thesis will be devoted to describing Rubin's method, with some easy generalisations, and to exploring the connection between Rubin's condition, the small index property, and the definability in $\mathrm{Aut}(\mathcal{M})$ of point stabilisers. We build a transitive structure where point stabilisers are definable, and which has the small index property, but does not have a weak $\forall\exists$ interpretation. We also use Rubin's method to obtain reconstruction for a countable set with two independent dense linear orders without endpoints living on it, and we extend Rubin's results about his so-called *simple* structures to a construction which amounts to superimposing two primitive simple structures living on the same set. We also construct a device, under certain conditions, for lifting weak $\forall\exists$ interpretations from definable closed normal subgroups to the full automorphism groups. The definability conditions on the subgroups involved can be proved via a further result, which is based on generic automorphisms.

In the rest of the thesis, we handle structures with relations of higher arity, and with functions.

In [17], it was shown that the random graph has the small index property. The method, using *ample homogeneous generic automorphisms*, rested on a lemma of Hrushovski on extensions of partial isomorphisms of finite graphs. In a series of papers, Herwig showed that variants of Hrushovski's lemma hold for many other homogeneous structures, such as universal k-hypergraphs, K_m-free graphs, and the *Henson digraphs*. It follows that these, too, have the small index property. In Chapter 2, using a Baire category argument and a small adaptation of Herwig's extension lemmas, we show that these structures also admit weak $\forall\exists$ interpretations (Theorems 2.2.2 and 2.3.12).

In the third chapter we obtain reconstruction results for the projective space $\mathrm{PG}(V)$, where V is a countably infinite dimensional vector space over a finite field, and for the projective symplectic, unitary and orthogonal spaces on V. The main theorem we prove is the following:

Theorem 1 *Let V be an $\aleph_0$-dimensional vector space over a finite field F_q, and* *let $\mathcal{M}$ be an ω-categorical structure with domain* $\mathrm{PG}(V)$ *and such that one of the following holds:*

1. $\mathrm{PGL}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma L}(V)$
2. $\mathrm{PSp}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma Sp}(V)$
3. $\mathrm{PU}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma U}(V)$
4. $\mathrm{PO}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma O}(V)$

where $\mathrm{PU}(V)$ *and* $\mathrm{PO}(V)$ *are the projective unitary and projective orthogonal groups respectively. Then $\mathcal{M}$ has a weak $\forall\exists$ interpretation*[1]*.*

These spaces are some of the *Lie geometries* which coordinate smoothly approximable structures of Cherlin and Hrushovski [8]. The small index property is known for the structures in Theorem 1 [11]. We also give a reconstruction result for the affine group $\mathrm{AGL}(V)$ acting on V, and for sufficiently large subgroups, by proving that V as an affine space is interpretable in $\mathrm{AGL}(V)$.

Overall, the thesis contains a clarification on the scope of applicability of Rubin's reconstruction method. Some important examples that the methods presented here cannot currently handle are the following:

- the random ordered graph $\langle \Omega, R, < \rangle$, where Ω is a countable set, R is a graph relation and $<$ is a linear order;
- Cherlin's semifree constructions [7];
- finite covers of Lie Geometries and, more generally, smoothly approximable structures.

In Section 1.2, an example (an equivalence relation with $\aleph_0$ classes of size 2) is given which does not have a weak $\forall\exists$ interpretation. A further task is to identify

[1] for the projective space and the projective unitary and orthogonal spaces, we obtain a weak $\forall\exists$ interpretation according to a more general definition than Rubin's original one

a generalisation of weak $\forall\exists$ interpretations which is robust under taking finite covers, and strong enough to give reconstruction results.

We set out some conventions on notation and terminology. Throughout the thesis, we shall work with ω-categorical structures, unless otherwise stated. We shall not distinguish between a structure $\mathcal{M}$ and its domain.

We shall use some notions and notation from infinite permutation group theory. The notation $\langle G, X\rangle$ is used for the group G acting on a set X. The image of $x \in X$ under $g \in G$ is denoted by x^g or by xg. If $\langle G, X\rangle$ is a permutation group and $x \in X$, then G_x denotes the stabiliser of x. If $C \subseteq X$, then G_C denotes the pointwise stabiliser of C,

$$G_C := \{g \in G : \forall c \in C,\ c^g = c\},$$

and $G_{\{C\}}$ denotes the setwise stabiliser

$$G_{\{C\}} := \{g \in G : \forall x,\ x \in C \leftrightarrow\ x^g \in C\}.$$

It will occasionally be convenient to denote G_C by $\mathrm{Stab}(C)$ or $\mathrm{Stab}_G(C)$. If $H \leq G$, then $\cos(G : H)$ is the set of right cosets of H; also,

$$\mathcal{N}_G(H) := \{g \in G : g^{-1}Hg = H\}$$

is the normaliser of H in G. For $g, h \in G$, $g \sim h$ means that g is conjugate to h. The centre of G is $Z(G) := \{h \in G : \forall g \in G\ hg = gh\}$.

A permutation group $\langle G, X\rangle$ is **transitive** if for any two $x, y \in X$ there is $g \in G$ with $x^g = y$. A transitive permutation group $\langle G, X\rangle$ is **primitive** if there are no non-trivial proper equivalence relations E on X such that, for all $x, y \in X$ and $g \in G$, $xEy \iff x^gEy^g$. The permutation group $\langle G, X\rangle$ is **regular** if it is transitive and for all $x \in X$ and $g \in G$, if $x^g = x$, then $g = \mathrm{id}$.

An ω-categorical structure $\mathcal{M}$ is said to have **no algebraicity** if for every finite subset $A \subseteq \mathcal{M}$ and $a \in \mathcal{M} \setminus A$, a is not algebraic over A.

Let $\langle G, X\rangle$ be an oligomorphic permutation group, let $X_0 \subseteq X$ be finite and $x \in X$. We say that x is **algebraic over** X_0 if $|\{x^g : g \in G_{X_0}\}| < \aleph_0$. Then

$\langle G, X \rangle$ **has no algebraicity** if for every finite $X_0 \subseteq X$ and $x \in X \setminus X_0$, x is not algebraic over X_0.

It is easy to see that if $\mathcal{M}$ is ω-categorical, $\mathcal{M}$ has no algebraicity if and only if $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M} \rangle$ has no algebraicity.

A countable structure is **homogeneous** if any isomorphism between finite substructures extends to an automorphism.

Chapter 1

Preliminaries

In this chapter we shall define Rubin's notion of weak $\forall\exists$ interpretations, state his main theorem and give a straightforward generalisation of his main result. We show how weak $\forall\exists$ interpretations compare to definibility of point stabilisers in the automorphism group. In [26], A. Singerman produces a weak $\forall\exists$ interpretation for a structure without the small index property. Here we build an example which has the small index property but for which no weak $\forall\exists$ interpretation can be found. Hence having a weak $\forall\exists$ interpretation is independent of the small index property (in the sense that neither condition implies the other). In Section 1.3 we use Rubin's method to treat an example which is not currently handled by small index. In the last two sections, we give methods to obtain new weak $\forall\exists$ interpretations from existing ones. One of these methods concerns Rubin's so-called *simple* structures.

1.1 The method

Rubin's main result gives a reconstruction criterion for the class of ω-categorical structures without algebraicity. Whenever such a structure $\mathcal{M}$ has a so called *weak* $\forall\exists$ *interpretation* and $\mathcal{N}$ is also ω-categorical without algebraicity, it is enough to know that $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$ as abstract groups in order to conclude that the permutation groups $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}\rangle$ and $\langle \mathrm{Aut}(\mathcal{N}), \mathcal{N}\rangle$ are isomorphic. Given an ω-categorical transitive structure $\mathcal{M}$, one selects a conjugacy class C in $\mathrm{Aut}(\mathcal{M})$ and a conjugacy invariant equivalence relation E on C that satisfies certain definability conditions in the language of groups, so that the permutation groups $\langle \mathrm{Aut}(\mathcal{M}), C/E\rangle$ and $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}\rangle$ are isomorphic. If such C and E can be found, $\mathcal{M}$ is said to have a *weak* $\forall\exists$ *interpretation*. Generally (but not necessarily) C consists of automorphisms having a single fixed point and E is "having the same fixed point". So an element of $\mathcal{M}$ is identified with the equivalence class of automorphisms that fix it. When $\mathcal{M}$ is not transitive, a weak $\forall\exists$ interpretation for $\mathcal{M}$ consists of a weak $\forall\exists$ interpretation for each orbit of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{M}$. Let us state the formal definitions (to be found in [25]).

Let G be a group acting transitively on the countable set $\mathcal{M}$, and let E be a G-invariant equivalence relation on $\mathcal{M}$, i.e. such that

$$\forall\ a, b \in \mathcal{M}, \forall\ g \in G\ aEb \Rightarrow a^g E b^g.$$

Then G has a natural action on the set of equivalence classes $\mathcal{M}/E$, that is, $(a/E)^g = (a^g)/E$, where a/E is the equivalence class of $a \in \mathcal{M}$.

Definition 1.1.1 *Let G be a group, and let $\bar{g} = \langle g_1, \ldots, g_n\rangle \in G^n$. Let $\phi(\bar{g}, x, y)$ be a formula in the language of groups with parameters $\bar{g}$. Let $C := g_1^G$. We say that ϕ is an $\forall\exists$* ***equivalence formula*** *for G if:*

- *ϕ is $\forall\exists$;*
- *Group theory $\vdash \forall \bar{u}(\phi(\bar{u}, x, y)$ is an equivalence relation on the conjugacy class of $u_1)$;*
- *for fixed $\bar{g}$, $\phi(\bar{g}, x, y)$ defines a conjugacy invariant equivalence relation on C.*

We shall write E^ϕ for the equivalence relation defined by ϕ.

Remark 1.1.2 Rubin's original definition of an $\forall\exists$ equivalence formula ϕ requires ϕ to define an equivalence relation that is conjugacy invariant in **all** groups. However, the proof of Lemma 1.1.9 below shows that Theorem 1.1.7 works under the weaker assumption that the equivalence relation defined by ϕ is conjugacy invariant in the given group.

Lemma 1.1.3 *Let G be a group, and let E be an equivalence relation on a conjugacy class of G such that E is defined with parameters by an existential formula $\phi(\bar{u}, x, y)$ in the language of groups. Let C denote the conjugacy class of u_1. Then*

$$\phi' \equiv \phi(\bar{u}, x, y) \wedge \forall \bar{u}(\phi(\bar{u}, x, y) \textit{ is an equivalence relation on } C)$$

is an $\forall\exists$ formula.

Proof Let $\phi(\bar{u}, x, y) \equiv \exists \bar{z}\ \phi_0(\bar{u}, \bar{z}, x, y)$. Then '$\forall \bar{u}(\phi(\bar{u}, x, y)$ is an equivalence relation on C)' is of the form

$$\forall \bar{u}(\forall x \exists \bar{z}\ \phi_0(\bar{u}, \bar{z}, x, x) \wedge \forall xy(\exists \bar{z}\ \phi_0(\bar{u}, \bar{z}, x, y) \rightarrow \exists \bar{z}\ \phi_0(\bar{u}, \bar{z}, y, x))$$

$$\wedge \forall xyw((\exists \bar{z}\ \phi_0(\bar{u}, \bar{z}, x, y) \wedge \exists \bar{z}\ \phi_0(\bar{u}, \bar{z}, y, w)) \rightarrow \exists \bar{z}\ \phi_0(\bar{u}, \bar{z}, x, w))).$$

The following hold for first order formulae C and D, where the variable y is not free in D:

1. $\exists x Cx \rightarrow D \dashv\vdash \forall y(Cy \rightarrow D)$;
2. $D \rightarrow \exists x Cx \dashv\vdash \exists y(D \rightarrow Cy)$;
3. $\forall z(C \wedge D) \dashv\vdash \forall z C \wedge \forall z D$;
4. $\exists y(Cy \wedge D) \dashv\vdash \exists y Cy \wedge D$.

By suitable changes of variables and using the equivalences 1.– 4. above, it is easy to see that ϕ' is in fact an $\forall\exists$ formula. $\square$

Definition 1.1.4 (Weak $\forall\exists$ interpretation, transitive case) *Let $\mathcal{M}$ be an ω-categorical structure such that* $\mathrm{Aut}(\mathcal{M})$ *acts transitively on $\mathcal{M}$. A* ***weak*** *$\forall\exists$*

interpretation *for $\mathcal{M}$ is a triple $\langle \phi(\bar{g}, x, y), \bar{g}, \tau \rangle$, where $\phi(\bar{g}, x, y)$ is an $\forall\exists$-equivalence formula for* $\mathrm{Aut}(\mathcal{M})$*, $\bar{g} \in \mathrm{Aut}(\mathcal{M})^n$, $C = g_1^{\mathrm{Aut}(\mathcal{M})}$, and τ is an isomorphism between the permutation groups $\langle \mathrm{Aut}(\mathcal{M}), C/E^{\phi} \rangle$ and $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M} \rangle$.*

Example 1.1.5 Consider the structure $\langle X, R \rangle$, where X is the set of subsets of Ω of size 2, for a countable Ω, and two subsets are joined by the relation R if they share exactly one element:

$$\{x, y\} R \{x', y'\} \iff |\{x, y\} \cap \{x', y'\}| = 1.$$

This structure is the **line graph** of the complete graph.

It can be proved that $\mathrm{Aut}(X) = \mathrm{Sym}(\Omega)$ (the inclusion $\mathrm{Sym}(\Omega) \subseteq \mathrm{Aut}(X)$ is immediate as R is clearly preserved by bijections). $\mathrm{Aut}(X)$ acts transitively on X. We choose:

- g to be any transposition of Ω (hence our $\bar{g}$ in the definition will be of length 1)
- E^{ϕ} to be the trivial equivalence relation
- τ to be the map which takes the 2-subset $\{a, b\}$ to the transposition (ab).

We have chosen a conjugacy class such that for each automorphism in the class there is exactly one 2-set which is fixed setwise but moved pointwise, so that a 2-subset in X is mapped to the automorphism swapping its elements. It is easy to see that $\mathrm{Aut}(X)$ acts on X isomorphically to its action on C by conjugation. Here we do not need to quotient the conjugacy class by an equivalence relation, as each point corresponds to exactly one element of C. In most cases, we need an equivalence relation which identifies automorphisms corresponding to the same point.

By the Ryll-Nardzewski theorem, if $\mathcal{M}$ is ω-categorical then $\mathcal{M}^n$ is partitioned into a finite number of orbits of $\mathrm{Aut}(\mathcal{M})$, for every $n \in \mathbb{N}$. In particular, $\mathcal{M}$ is partitioned into finitely many orbits on $\mathcal{M}$, corresponding to 1-types, and

$\mathrm{Aut}(\mathcal{M})$ acts transitively on each of them. We can thus extend the definition of a weak $\forall\exists$ interpretation to the general case when $\mathcal{M}$ is not transitive.

Definition 1.1.6 *[Weak $\forall\exists$ interpretation] Let $\mathcal{M}$ be an ω-categorical structure, let $P_1, \ldots, P_n$ be the orbits of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{M}$. A **weak** $\forall\exists$ **interpretation** for $\mathcal{M}$ is an object $\langle \vec{\phi}, \vec{g}, \vec{\tau} \rangle$, where $\vec{\phi} = \langle \phi_1, \ldots, \phi_n \rangle$ are $\forall\exists$ equivalence formulae, $\vec{g} = \langle \vec{g}^1, \ldots, \vec{g}^n \rangle$ are tuples of elements of $\mathrm{Aut}(\mathcal{M})$, $\vec{\tau} = \langle \tau_1, \ldots, \tau_n \rangle$ are maps such that each triple $\langle \phi_i, \vec{g}^i, \tau_i \rangle$ is a weak $\forall\exists$ interpretation for the structure induced on P_i.*

We can now state Rubin's main result.

Theorem 1.1.7 (Rubin, 87) *Let $\mathcal{M}$ be an ω-categorical structure without algebraicity, and suppose $\mathcal{M}$ has a weak $\forall\exists$-interpretation. Suppose $\mathcal{N}$ is also ω-categorical, without algebraicity and such that $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}(\mathcal{N})$ are isomorphic as pure groups. Then $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M} \rangle \cong \langle \mathrm{Aut}(\mathcal{N}), \mathcal{N} \rangle$, that is, $\mathcal{M}$ and $\mathcal{N}$ are bi-definable.*

A trivial generalisation of Rubin's proof allows us to extend the definition of weak $\forall\exists$ interpretation to the case where the conjugacy class involved is in fact a conjugacy class on a tuple, i.e. $C = \langle g_1, \ldots, g_k \rangle^{\mathrm{Aut}(\mathcal{M})}$ for $\langle g_1, \ldots, g_k \rangle \in \mathrm{Aut}(\mathcal{M})^k$. The modifications are needed in Rubin's pivotal Lemma 2.6 in [25]. With our version of Rubin's 2.6, the proofs of 1.1.7 and 1.1.10 below go through unchanged, so that 1.1.7 and 1.1.10 hold with this more general definition of weak $\forall\exists$ interpretation. From now on, by "weak $\forall\exists$ interpretation" we shall mean a weak $\forall\exists$ interpretation in this more general sense. Rubin's original definition is the special case $k = 1$.

Definition 1.1.8 *An equivalence relation on a set of n-tuples X is **non-degenerate** if there is no $I \subsetneq \{1, \ldots, n\}$ such that for every $\bar{a} = \langle a_1, \ldots, a_n \rangle$ and $\bar{b} = \langle b_1, \ldots, b_n \rangle$ in X, if for every $i \in I$ $a_i = b_i$, then $\bar{a} E \bar{b}$.*

Lemma 1.1.9 *Let $\mathcal{M}$ be ω-categorical, and let $\langle f_1, \ldots, f_n, f_{n+1}, \ldots, f_m\rangle \in \mathrm{Aut}(\mathcal{M})^m$. Suppose that $\phi(u_1, \ldots, u_n, u_{n+1}, \ldots, u_m, x_1, \ldots, x_n, y_1, \ldots, y_n)$ is an $\forall\exists$ equivalence formula for $\mathrm{Aut}(\mathcal{M})$, so that the equivalence relation E^ϕ defined by ϕ on $C = \langle f_1, \ldots, f_n\rangle^{\mathrm{Aut}(\mathcal{M})}$ is conjugacy invariant. Suppose further that E^ϕ is such that $|C/E^\phi| = \aleph_0$.*
Then there are, for some i, a complete i-type t of $\mathcal{M}$, a non-degenerate 0-definable equivalence relation R on the set $\mathcal{M}_t$ of realisations of t, and a bijection $\tau : \mathcal{M}_t/R \to C/E^\phi$ which induces an isomorphism

$$\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}_t/R\rangle \cong \langle \mathrm{Aut}(\mathcal{M}), C/E^\phi\rangle.$$

Proof We sketch Rubin's argument for [25] 2.6, extended to a conjugacy class on n-tuples. Note that among the parameters $\bar{f}$ appearing in ϕ, $f_1, \ldots, f_n$ define the conjugacy class C, and $f_{n+1}, \ldots, f_m$ are extra parameters.

Let v be a real encoding the relations of $\mathcal{M}$, $\{\bar{g}_i : i \in \omega\}$ be a sequence of representatives of all the equivalence classes of E^ϕ. Choose the $\bar{g}_i$ so that for every $i \neq j$, $\neg \bar{g}_i E^\phi \bar{g}_j$, and let r be a real encoding $\{\bar{g}_i : i \in \omega\}$. Since ϕ is $\forall\exists$, there is a Π^1_2-formula $\psi^*(v, r, \bar{f})$ which says that for all $\bar{g} \in C$ there is $i \in \omega$ such that $\bar{g} E^\phi \bar{g}_i$. By Shoenfield's theorem, $\forall\exists$ statements are absolute in generic extensions of the universe. So for every generic extension $V[G]$, $\psi^*(v, r, \bar{f})$ holds in $V[G]$. We build such an extension. Let $\langle \mathcal{M}^*, f_1^*, \ldots, f_n^*\rangle \cong \langle \mathcal{M}, f_1, \ldots, f_n\rangle$. Define

$$P := \{h : h \text{ is a finite partial isomorphism } \mathcal{M}^* \to \mathcal{M}\},$$

and $h_2 > h_1$ if $h_2 \supseteq h_1$. If G is a generic filter for P, then $h_G := \bigcup G$ is an isomorphism between $\mathcal{M}^*$ and $\mathcal{M}$. Let $f^*_{i,G} := (f_i^*)^{h_G}$. The idea is to select $h_0 \in P$ which forces $\bar{f}^*_G$ to be in a certain equivalence class. Then the type t that we look for will be the type of the range of h_0, and two tuples in the type are equivalent under R if and only if they force $\bar{f}^*_G$ to be in the same E^ϕ equivalence class.

In $V[G]$ the tuple $\langle f^*_{1,G}, \ldots, f^*_{n,G}\rangle$ is conjugate to $\langle f_1, \ldots, f_n\rangle$, so for some $i_0 \in \omega$,

$\bar{f}^*_G E^\phi \bar{g}_{i_0}$. Let h_0 be the least element of P such that

$$h_0 \Vdash \bar{f}^*_G E^\phi \bar{g}_{i_0}.^1$$

For all $k \in \mathrm{Aut}(\mathcal{M})$ define $\alpha_k \in \mathrm{Aut}(P, <)$ as follows: $\alpha_k(h) := k \circ h$. Rubin shows that for any formula χ in the language of groups,

$$h \Vdash \text{“}\chi(\bar{f}^*_G, l_1, \ldots, l_m) \text{ holds in } \mathrm{Aut}(\mathcal{M})\text{”}$$

if and only if

$$\alpha_h(k) \Vdash \text{“}\chi(\bar{f}^*_G, l_1^k, \ldots, l_m^k) \text{ holds in } \mathrm{Aut}(\mathcal{M})\text{”}.$$

Without loss of generality we can assume $i_0 = 0$. Since $\neg \bar{g}_0 E^\phi \bar{g}_1$, $\mathrm{Aut}(\mathcal{M}) \models \neg\phi(\bar{f}, \bar{g}_0, \bar{g}_1)$. This can be translated to a Σ^1_2 statement, so, again by Shoenfield's theorem,

$$\Vdash_P \mathrm{Aut}(\mathcal{M}) \models \neg\phi(\bar{f}, \bar{g}_0, \bar{g}_1). \qquad (*)$$

Now pick $k \in \mathrm{Aut}(\mathcal{M})$ such that $\bar{g}_0^k = \bar{g}_1$. By $h_0 \Vdash \bar{f}^*_G E^\phi \bar{g}_0$, by $(*)$ and by the equivalence above, we get that $k \circ h_0 \Vdash \bar{f}^*_G E^\phi \bar{g}_1$. But ϕ defines an equivalence relation, so h_0 and $k \circ h_0 = \alpha_k(h_0)$ are contradictory, so we can conclude that $|h_0| > 0$.

We define the required type t as follows: let $h_0 = \{\langle a_0, b_0^0\rangle, \ldots, \langle a_i, b_i^0\rangle\}$ and $\bar{b}^0 := \langle b_0^0, \ldots, b_i^0\rangle$, so that the b_j^0 are distinct. Let $t = \mathrm{tp}(\bar{b}^0)$. Define an equivalence relation R on $\mathcal{M}_t$ as follows: for $\bar{b} = \langle b_0, \ldots . b_i\rangle \in \mathcal{M}_t$ let $h_{\bar{b}} := \{\langle a_0, b_0\rangle, \ldots, \langle a_i, b_i\rangle\}$. Then $\bar{b}^1 R \bar{b}^2 \in \mathcal{M}_t$ if and only if there is $j \in \omega$ such that

$$h_{\bar{b}^1} \Vdash \mathrm{Aut}(\mathcal{M}) \models \phi(\bar{f}, \bar{f}^*_G, \bar{g}_j) \text{ and } h_{\bar{b}^2} \Vdash \mathrm{Aut}(\mathcal{M}) \models \phi(\bar{f}, \bar{f}^*_G, \bar{g}_j).$$

As in [25], R is invariant under the action of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{M}$. Moreover, it is non-degenerate and 0-definable.

Finally, define $\tau : \mathcal{M}_t/R \to C/E^\phi$ by

$$\tau(\bar{b}/R) = \bar{g}_i/E^\phi, \text{ where } h_{\bar{b}} \Vdash \bar{f}^*_G E^\phi \bar{g}_i.$$

[1]What makes Rubin's proof work for a conjugacy class on tuples is that the condition $\bar{f}^*_G E^\phi \bar{g}_{i_0}$ is still forced by a **single** element of P

Then τ induces an isomorphism between $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}_t/R\rangle$ and $\langle \mathrm{Aut}(\mathcal{M}), C/E\rangle$: let $\bar{b}/R$, $\bar{c}/R \in \mathcal{M}_t/R$ and $k \in \mathrm{Aut}(\mathcal{M})$ be such that $(\bar{b}/R)^k = \bar{c}/R$. Suppose $h_{\bar{b}} \Vdash \bar{f}^*_G E^\phi \bar{g}_i$, so that $\tau(\bar{b}/R) = \bar{g}_i/E^\phi$, and let $\bar{g}_i^k E^\phi \bar{g}_j$. We need to show that $\tau(\bar{c}/R) = \bar{g}_i^k/E^\phi = \bar{g}_j/E^\phi$, i.e. that $h_{\bar{c}} \Vdash \bar{f}^*_G E^\phi \bar{g}_j$. We may assume $\bar{c} = \bar{b}^k$. Then $h_{\bar{b}} \Vdash \bar{f}^*_G E^\phi \bar{g}_i$ implies $k \circ h_{\bar{b}} = h_{\bar{c}} \Vdash \bar{f}^*_G E^\phi \bar{g}_i^k$. Since $\bar{g}_i^k E^\phi \bar{g}_j$, we get that $h_{\bar{c}} \Vdash \bar{f}^*_G E^\phi \bar{g}_j$, as required. $\square$

If we drop the assumption of absence of algebraicity in 1.1.7, a weak $\forall\exists$-interpretation plays the same role as the small index property as far as reconstruction is concerned ([25], p. 227):

Proposition 1.1.10 *Let $\mathcal{M}$, $\mathcal{N}$ be ω-categorical and let $\mathcal{M}$ have a weak $\forall\exists$ interpretation. Then: if* $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$*, $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable.*

Proof Suppose $\mathcal{M}$ has a weak $\forall\exists$ interpretation

$$\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}\rangle \cong \langle \mathrm{Aut}(\mathcal{M}), \bigcup_{i=1}^{n} C_i/E_i\rangle \qquad (*)$$

where, for $i = 1, \ldots, n$, C_i is a conjugacy class in $\mathrm{Aut}(\mathcal{M})^{k_i}$ for some $k_i \in \mathbb{N}$, and E_i is a conjugacy invariant equivalence relation on C_i, defined by the $\forall\exists$ equivalence formula ϕ_i in the language of groups.

Suppose $\alpha : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{N})$ is an isomorphism. Let $C_i' := \alpha(C_i)$ for $i = 1, \ldots, n$. Then, by isomorphism with $\mathrm{Aut}(\mathcal{M})$, $|C_i'/E_i| = \aleph_0$ for all $i = 1, \ldots, n$. By 1.1.9 above (i.e. Rubin's lemma 2.6 in [25]), there are complete types $t_1, \ldots, t_n$ on $\mathcal{N}$ and 0-definable equivalence relations R_i on the set $\mathcal{N}_{t_i}$ of realisations of t_i, for $i = 1, \ldots, n$, such that

$$\langle \mathrm{Aut}(\mathcal{N}), \bigcup_{i=1}^{n} C_i'/E_i\rangle \cong \langle \mathrm{Aut}(\mathcal{N}), \bigcup_{i=1}^{n} \mathcal{N}_{t_i}/R_i\rangle.$$

By $(*)$, and by $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$, we get that

$$\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}\rangle \cong \langle \mathrm{Aut}(\mathcal{N}), \bigcup_{i=1}^{n} \mathcal{N}_{t_i}/R_i\rangle \qquad (**).$$

We now want to show that $\langle \mathrm{Aut}(\mathcal{N}), \mathcal{N}\rangle$ and $\langle \mathrm{Aut}(\mathcal{N}), \bigcup_{i=1}^{n} \mathcal{N}_{t_i}/R_i\rangle$ have the same open subgroups. Then, via (**), we can deduce that $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}(\mathcal{N})$ are isomorphic as topological groups, i.e. that $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable.

By (**) the action $\langle \mathrm{Aut}(\mathcal{N}), \bigcup_{i=1}^{n} \mathcal{N}_{t_i}/R_i\rangle$ is closed, oligomorphic and faithful. By Lemma 2.10 in [25], we get that $\mathcal{N}^{\mathrm{eq}} \subseteq \mathrm{dcl}(\bigcup_{i=1}^{n} \mathcal{N}_{t_i}/R_i)$ hence $\mathcal{N} \subseteq \mathrm{dcl}(\bigcup_{i=1}^{n} \mathcal{N}_{t_i}/R_i)$. Therefore, if $\bar{a} \in \mathcal{N}^l$ then $\bar{a} \in \mathrm{dcl}(\bar{b})$ for some tuple $\bar{b}$ of $\bigcup_{i=1}^{n} \mathcal{N}_{t_i}/R_i$. Then clearly $\mathrm{Aut}(\mathcal{N})_{\bar{b}} \subseteq \mathrm{Aut}(\mathcal{N})_{\bar{a}}$, so $\mathrm{Aut}(\mathcal{N})_{\bar{a}}$ is open in $\langle \mathrm{Aut}(\mathcal{N}), \bigcup_{i=1}^{n} \mathcal{N}_{t_i}/R_i\rangle$. Since $\bigcup_{i=1}^{n} \mathcal{N}_{t_i}/R_i$ is (trivially) interpretable in $\mathcal{N}$, a similar argument shows that $\mathrm{Aut}(\mathcal{N})_{\bar{b}}$ is open in $\langle \mathrm{Aut}(\mathcal{N}), \mathcal{N}\rangle$, as required. □

A consequence of the existence of a weak $\forall\exists$ interpretation, interesting in its own right, is that an ω-categorical structure is interpretable with parameters in its automorphism group.

Proposition 1.1.11 *If $\mathcal{M}$ has a weak $\forall\exists$ interpretation, then $\mathcal{M}$ is interpretable with parameters in* $\mathrm{Aut}(\mathcal{M})$.

Proof We want to show that there exist:

- a definable subset $D \subseteq \mathrm{Aut}(\mathcal{M})^n$, for some $n \in \mathbf{N}$,
- a definable equivalence relation E on D and
- a bijection $\alpha : \mathcal{M} \to D/E$

such that for any 0-definable m-ary relation R on $\mathcal{M}$ there is a definable mn-ary relation $\hat{R}$ on $\mathrm{Aut}(\mathcal{M})$ such that

$$\mathcal{M} \models R(b_1, \ldots, b_m)$$

if and only if for any $\bar{a}_i \in \alpha(b_i)$, $i = 1, \ldots, m$

$$\mathrm{Aut}(\mathcal{M}) \models \hat{R}(\bar{a}_1, \ldots, \bar{a}_m).$$

Suppose that $\mathcal{M}$ is transitive, and let $\langle \phi, \vec{g}, \tau\rangle$ be a weak $\forall\exists$ interpretation for $\mathcal{M}$. Then set

– $D := C$, the conjugacy class of $\langle g_1, \ldots, g_n \rangle$,

– $E := E^\phi$, the conjugacy invariant equivalence relation defined by ϕ, and

– $\alpha := \tau^{-1}$.

Take any 0-definable relation R on $\mathcal{M}^n$. Then R is the union of finitely many orbits of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{M}^n$. Therefore $\tau(R)$ is the union of finitely many orbits of $\mathrm{Aut}(\mathcal{M})$ on $(C/E)^n$. An orbit in $(C/E)^n$ is a set $\{\bar{g}^h/E : h \in \mathrm{Aut}(\mathcal{M})\}$. Conjugacy is definable in the language of groups, and E is definable by hypothesis. Therefore orbits in $(C/E)^n$ are definable in $\mathrm{Aut}(\mathcal{M})$.

Essentially the same argument works when $\mathcal{M}$ is not transitive. $\square$

1.2 Weak $\forall\exists$ interpretations and definability of point stabilisers

Let $\mathcal{M}$ be the structure $\langle \Omega, B \rangle$, where Ω is a countable set and B is an equivalence relation having $\aleph_0$ equivalence classes, each containing exactly two elements. This structure is ω-categorical and ω-stable, hence it has the small index property (see [17]), so it can be recovered from its automorphism group.

We can think of the domain of $\mathcal{M}$ as the cartesian product $\Delta \times \Omega$, where $\Delta = \{0, 1\}$. Then $\mathrm{Aut}(\mathcal{M})$ is the wreath product C_2 Wr $\mathrm{Sym}(\Omega)$, i.e. the semidirect product

$$\prod_{n \in \omega} (C_2)_n \rtimes \mathrm{Sym}(\Omega).$$

Note that:

- $\mathrm{Sym}(\Omega)$ acts on $\prod_{n \in \omega} (C_2)_n$ by $\bar{c}^\sigma = \bar{c}'$, where $c'_n = c_{n\sigma^{-1}}$, so σ acts by permuting the components of $\bar{c}$;

- multiplication in C_2 Wr $\mathrm{Sym}(\Omega)$ is defined in the usual way:

$$(\bar{c}_1, \sigma_1)(\bar{c}_2, \sigma_2) = (\bar{c}_1 \bar{c}_2^{\,\sigma_1^{-1}},\ \sigma_1 \sigma_2);$$

- C_2 Wr Sym(Ω) acts on $\Delta \times \Omega$ by

$$(\delta, n)^{(\bar{c},\sigma)} = (\delta^{c_n}, n^\sigma).$$

  This amounts to shifting one equivalence class to another while fixing or flipping the elements within each equivalence class.

We now show that no weak $\forall\exists$ interpretation for $\langle \Omega, B\rangle$ can be found. The argument we use will appear again in Lemma 3.1.1 below.

Proposition 1.2.1 *Let $\mathcal{M}$ be an ω-categorical structure such $Z(\mathrm{Aut}(\mathcal{M})) \neq \{\mathrm{id}\}$. Then $\mathcal{M}$ does not have a weak $\forall\exists$ interpretation.*

Proof Let $h \in Z(\mathrm{Aut}(\mathcal{M}))$, $h \neq \mathrm{id}$. Let $m \in \mathcal{M}$ be such that $m^h \neq m$, and let $P := \{m^g : g \in \mathrm{Aut}(\mathcal{M})\}$. Suppose for a contradiction that $\langle \phi, \vec{g}, \tau\rangle$ is a weak $\forall\exists$ interpretation for $\langle \mathrm{Aut}(\mathcal{M}), P\rangle$. Let $C = \langle g_1, \ldots, g_n\rangle^{\mathrm{Aut}(\mathcal{M})}$, and let $m = \tau(\langle k_1, \ldots, k_n\rangle/E^\phi)$. Then $(\langle k_1, \ldots, k_n\rangle/E^\phi)^h = \langle k_1, \ldots, k_n\rangle/E^\phi$, so h fixes $\langle k_1, \ldots, k_n\rangle/E^\phi$ but it does not fix $m = \tau(\langle k_1, \ldots, k_n\rangle/E^\phi)$, which is a contradiction. $\square$

Corollary 1.2.2 *The structure $\mathcal{M} = \langle \Omega, B\rangle$ does not have a weak $\forall\exists$ interpretation.*

Proof Let $h = (\bar{c}, \mathrm{id}_{\mathrm{Sym}(\Omega)}) \in C_2$ Wr Sym(Ω), where $c_i = (01)$ for all $i \in \omega$. Then $h \in Z(C_2$ Wr Sym$(\Omega))$, and $h \neq \mathrm{id}$, so, by 1.2.1, the claim follows. $\square$

1.2.1 Definability of point stabilisers

The structure $\langle \Omega, R\rangle$ is also interesting as an example where point stabilisers are definable in the automorphism group. As observed by Lascar [22], it follows easily from the definition that the existence of a weak $\forall\exists$ interpretation implies $\forall\exists$ definability of point stabilisers:

Fact 1.2.3 *Let $\mathcal{N}$ be an ω-categorical structure with a weak $\forall\exists$ interpretation $\langle \phi, \bar{g}, \tau \rangle$. Let $G = \mathrm{Aut}(\mathcal{N})$. Then any point stabiliser G_a, $a \in \mathcal{N}$, is $\forall\exists$ definable in $\langle G, \cdot \rangle$ with parameters in G.*

Proof For any G_a, $a \in \mathcal{N}$, pick $\bar{k} \in C = \langle g_1, \ldots, g_n \rangle^G$ such that $\tau(\bar{k}/E_\phi) = a$, where E_ϕ is the equivalence relation defined by ϕ. By the isomorphism of the action of $\mathrm{Aut}(\mathcal{N})$ on $\mathcal{N}$ and on C/E_ϕ, $h \in G_a$ if and only if $h \in G_{\bar{k}/E_\phi}$. But $G_{\bar{k}/E_\phi}$ is definable in the language of groups with parameters $\bar{g}, \bar{k}$:

$$\begin{aligned} h \in G_{\bar{k}/E_\phi} &\iff \bar{k}^h/E_\phi = \bar{k}/E_\phi \\ &\iff \bar{k}^h E_\phi \bar{k} \\ &\iff \phi(\bar{g}, \bar{k}, \bar{k}^h). \end{aligned}$$

Note that G_a is $\forall\exists$-definable because ϕ is $\forall\exists$. $\square$

The following proposition shows that in $\mathcal{M} = \langle \Omega, B \rangle$, point stabilisers are indeed $\forall\exists$ definable. Together with Corollary 1.2.2 this shows that this definability condition is necessary but not sufficient for the existence of a weak $\forall\exists$ interpretation. In the next section, we shall see a special case where a slightly stronger definability condition on point stabilisers in fact implies a weak $\forall\exists$ interpretation.

We shall need the following result by Bertram:

Theorem 1.2.4 (Bertram, 1971) *Let ς be any permutation of the countable set Ω such that ς has infinite support. Then every permutation of Ω is a product of four permutations, each conjugate to ς.*

Proof [3]. $\square$

Proposition 1.2.5 *For any $m \in \mathcal{M} = \langle \Omega, B \rangle$, G_m is $\exists$ definable with parameters in the language of groups.*

Proof In this structure the stabiliser of a point m coincides with the pointwise stabiliser of the B block where m lies. So we are really aiming at defining pointwise stabilisers of B blocks.

We start by defining the setwise stabiliser of an equivalence class of B. Let $G = C_2 \text{ Wr Sym}(\Omega)$, let $\{B_n : n \in \mathbb{N}\}$ list the B-classes, and let $G_{\{B_n\}} = \{g \in G : B_n^g = B_n\}$. Let $h \in G$ be the transposition flipping over the two elements in B_n and fixing everything else, i.e.

$$h = (\bar{d}, \mathrm{id}_{\mathrm{Sym}(\Omega)}) \text{ where } \bar{d} = (\mathrm{id}_{C_2}, \ldots, (01),\ \mathrm{id}_{C_2}, \ldots).$$

Then $G_{\{B_n\}} = \{g \in G : hh^g = \mathrm{id}\}$, so $G_{\{B_n\}}$ is definable in the language of groups with parameter h by a quantifier free formula.

We show first that $G_{\{B_n\}} \subseteq \{g \in G : hh^g = \mathrm{id}\}$. For any $g \in G_{\{B_n\}}$ either

1. g swaps the two elements of B_n, i.e. $g = ((c_0,\ \ldots,\ c_{n-1},\ (01),\ c_{n+1}, \ldots),\ \sigma)$, where $n^\sigma = n$, or

2. g fixes B_n pointwise, i.e. $g = ((c_0,\ \ldots,\ c_{n-1},\ \ \mathrm{id}_{C_2},\ c_{n+1}, \ldots),\ \sigma)$, where $n^\sigma = n$.

Clearly, in either case conjugation by g has (on B_n) only the effect of swapping 0 and 1 twice, hence $h^g = h$ as required (note that h has order 2).

Next we show $\{g \in G : hh^g = \mathrm{id}\} \subseteq G_{\{B_n\}}$. Let $g = (\bar{c}, \sigma)$ be such that $h^g = h$. We want to show that g stabilises B_n setwise. This happens when $n^\sigma = n$. Now:

$$\begin{aligned} h^g &= ((\bar{c}^{-1}\bar{d}\bar{c})^\sigma,\ \sigma^{-1}\mathrm{id}_{\mathrm{Sym}(\Omega)}\ \sigma) \\ &= ((c_0^{-1}\ \mathrm{id}_{C_2}\ c_0, \ldots,\ c_n^{-1}\ (01)\ c_n, \ldots)^\sigma,\ \mathrm{id}_{\mathrm{Sym}(\Omega)}) \\ &= ((\mathrm{id}_{C_2}, \ldots, (01),\ \mathrm{id}_{C_2}, \ldots),\ \mathrm{id}_{\mathrm{Sym}(\Omega)}) \end{aligned}$$

hence $h^g = h$ if and only if

$$(\mathrm{id}_{C_2},\ \ldots,\ c_n^{-1}(01)c_n,\ \mathrm{id}_{C_2}, \ldots)^\sigma = (\mathrm{id}_{C_2},\ \ldots,\ (01),\ \mathrm{id}_{C_2},\ \ldots)$$

and this requires $n^\sigma = n$ (otherwise σ swaps some id_{C_2} with (01), and equality no longer holds).

Now $G_{\{B_n\}}$ is definable by the quantifier free formula

$$\chi(x,\ h)\ \equiv\ hh^x = \mathrm{id}$$

and we shall use this fact to define the pointwise stabiliser G_{B_n}. Let $u \in G_{B_n}$ flip $\aleph_0$ blocks and fix $\aleph_0$ blocks pointwise (without moving any blocks at all), that is, $u = (\bar{d}, \mathrm{id}_{\mathrm{Sym}(\Omega)})$, where $|\{d_i : d_i = \mathrm{id}_{C_2}\}| = |\{d_j : d_j = (01) \in C_2\}| = \aleph_0$. Let $v = (\mathrm{id}, \rho) \in G_{\{B_n\}}$, where $\rho \in \mathrm{Sym}(\Omega)$ is any permutation having infinite support.

Claim: every $g \in G_{B_n}$ can be written as a product $v^{g_1}v^{g_2}v^{g_3}v^{g_4}u^{g_5}u^{g_6}$, with $g_5, g_6 \in G_{B_n}$.

Let $g = (\bar{c}, \sigma)$. By 1.2.4, σ can be written as a product $\rho^{\tau_1}\rho^{\tau_2}\rho^{\tau_3}\rho^{\tau_4}$ for some $\tau_i \in \mathrm{Sym}(\Omega)$. Choose $g_i = (\mathrm{id}_{\prod_{n\in\omega}(C_2)_n},\ \tau_i)$ for $i = 1, \ldots, 4$. Then

$$k := g^{-1}v^{g_1}v^{g_2}v^{g_3}v^{g_4}$$

fixes each block setwise and B_n pointwise.

Write $k = (\bar{b}, \mathrm{id}_{\mathrm{Sym}(\Omega)})$, and observe that k will:

1. fix pointwise a possibly infinite number of blocks, say $\{B_i : i \in I\}$ (so $n \in I$);

2. flip a possibly infinite number of blocks, say $\{B_j : j \in J\}$.

Choose $h_1 = (\bar{d}^1, \mathrm{id}_{\mathrm{Sym}(\Omega)}),\ h_2 = (\bar{d}^2, \mathrm{id}_{\mathrm{Sym}(\Omega)}) \in G_{B_n}$ so that they both flip $\aleph_0$ blocks, fix $\aleph_0$ blocks and move no other blocks at all. Then $h_1,\ h_2$ are conjugate to u by permutations g_5^{-1}, g_6^{-1}, where $g_i = (\mathrm{id}_{\prod_{n\in\omega}(C_2)_n},\ \tau_i)$ with $p\tau_i = q \iff d^j_p = d_q \in C_2$ for $i = 5, 6$ and $j = 1, 2$ (i.e. τ_i moves the indexes of $\bar{d}^j$ so that $\bar{d}^j$ gets rearranged as $\bar{d}$). It is easy to see that the τ_i can be chosen such that $n\tau_i = n$, so that $g_5, g_6 \in G_{\{B_n\}}$. The product h_1h_2 fixes a block B_x pointwise if and only if $B_x \subseteq \mathcal{M} \setminus \mathrm{Supp}(h_1) \triangle \mathrm{Supp}(h_2)$, i.e. if B_x is either fixed by both h_1 and h_2, or fixed by both. Moreover, $h_1,\ h_2$ can be chosen so that

$$B_x\ \subseteq\ \mathrm{Supp}(h_1) \triangle \mathrm{Supp}(h_2)\ \iff\ x \in J$$

(so $B_x \in \mathcal{M} \setminus \mathrm{Supp}(h_1) \triangle \mathrm{Supp}(h_2) \iff x \in I$). Hence the product h_1h_2 will be such that $kh_1h_2 = \mathrm{id}$, that is

$$g^{-1}v^{g_1}v^{g_2}v^{g_3}v^{g_4}u^{g_5}u^{g_6} = \mathrm{id}.$$

Therefore g can be written in the form $v^{g_1}v^{g_2}v^{g_3}v^{g_4}u^{g_5}u^{g_6}$, with $v^{g_1}v^{g_2}v^{g_3}v^{g_4}$ and $g_5, g_6 \in G_{\{B_n\}}$.

Conversely, any product $v^{g_1}v^{g_2}v^{g_3}v^{g_4}$ is of the form $(\mathrm{id}_{\prod_{n\in\omega}(C_2)_n}, \sigma)$, so if it fixes B_n setwise, it also fixes it pointwise. Also, $u \in G_{B_n}$ and $g_5, g_6 \in G_{\{B_n\}}$ implies that $u^{g_5}, u^{g_6} \in G_{B_n}$. Hence any product of the form $v^{g_1}v^{g_2}v^{g_3}v^{g_4}u^{g_5}u^{g_6}$, with $v^{g_1}v^{g_2}v^{g_3}v^{g_4}$ and $g_5, g_6 \in G_{\{B_n\}}$ is in G_{B_n}.

Therefore G_{B_n} is defined by the existential formula $\psi(x, h, u, v)$

$$\exists y_1y_2y_3y_4y_5y_6\ (\chi(v^{y_1}v^{y_2}v^{y_3}v^{y_4}) \wedge \chi(y_5) \wedge \chi(y_6) \wedge x = v^{y_1}v^{y_2}v^{y_3}v^{y_4}u^{y_5}u^{y_6}). \quad \square$$

1.2.2 The primitive case

If $\mathcal{M}$ is a primitive ω-categorical structure, we show that if point stabilisers are existentially definable, then $\mathcal{M}$ has a weak $\forall\exists$ interpretation.

If $\mathcal{M}$ is primitive, then it is also transitive, so for $a \in \mathcal{M}$ we know that

$$\langle G, \mathcal{M}\rangle \cong \langle G, \cos\,(G : G_a)\rangle.$$

Recall that G acts on $\cos\,(G : G_a)$ by right multiplication. Also, if

$$H^G = \{g^{-1}Hg\ :\ g \in G\},$$

G acts on H^G by conjugation.

Fact 1.2.6 *Let $\langle G, X\rangle$ be a group action. The following are equivalent:*

- *$\langle G, X\rangle$ is primitive;*
- *for every $a \in X$, the stabiliser G_a is a maximal subgroup of G.*

Proof [4], Theorem 4.7. $\square$

Lemma 1.2.7 *Let $\langle G, X\rangle$ be a primitive action, and let $a \in X$. Then either $\mathcal{N}_G(G_a) = G_a$ or $\mathcal{N}_G(G_a) = G$. In particular, if $X = \mathcal{M}$, where $\mathcal{M}$ is an ω-categorical structure and $G = \mathrm{Aut}(\mathcal{M})$, point stabilisers in $\mathcal{M}$ are self normalising.*

Proof This is a direct consequence of Fact 1.2.6 and of the fact that $G_a \leq \mathcal{N}_G(G_a) \leq G$. The only case where $\mathcal{N}_G(G_a) = G$ is when $G_a \lhd G$, which only happens for C_p acting regularly on itself: for $a \in C_p$, $(C_p)_a = \{id\} \lhd C_p$. $\square$

The following lemma is central to our construction.

Lemma 1.2.8 *If G acts primitively on $\mathcal{M}$ and $H = G_a$ for some $a \in \mathcal{M}$ then the action of G on cos $(G : H)$ (by right multiplication) is isomorphic to its action on H^G (by conjugation).*

Proof Let α : cos $(G : H) \to H^G$ be defined by $\alpha(Hg) := H^g = g^{-1}Hg$. Then:

1. α is clearly surjective

2. α is injective, for:
   $\alpha(Ha) = \alpha(Hb) \Rightarrow H^a = H^b \Rightarrow H^{ab^{-1}} = H \Rightarrow ab^{-1} \in H$ (as H is self-normalizing by hypothesis) $\Rightarrow Ha = Hb$, as required.

3. α is a G-morphism:

$$\alpha((Hg)^k) = \alpha(Hgk) = H^{gk} = (H^g)^k = (\alpha(Hg))^k. \qquad \square$$

From the isomorphism in the lemma above we get

Proposition 1.2.9 *Suppose $\mathcal{M}$ is an ω-categorical structure such that $G = \mathrm{Aut}(\mathcal{M})$ acts primitively on $\mathcal{M}$. Let $a \in \mathcal{M}$ be such that the point stabiliser $H = G_a$ is $\exists$ definable in G in the language of groups by $\phi(x,\ \bar{b})$. Then $\mathcal{M}$ has a weak $\forall\exists$ interpretation.*

Proof If H is definable from $\bar{b}$, then H^g is definable from $\bar{b}^g$, for if $G \models \phi(h,\ \bar{b})$ then $G \models \phi(h^g,\ \bar{b}^g)$, since given any g, conjugation by g is an automorphism of G. So let $C = \bar{b}^G$ be the conjugacy class of the tuple $\bar{b}$. Define a relation E on C by identifying two tuples if they define the same conjugate of H:

$$\bar{c}E\bar{d} \iff \phi(G, \bar{c}) = \phi(G, \bar{d}).$$

Then:

1. E is an equivalence relation, and the formula defining E defines an equivalence relation in any group;
2. E is $\forall\exists$ definable, because ϕ is $\exists$ definable by hypothesis;
3. E is conjugacy invariant: $\bar{c}E\bar{d} \iff \phi(G, \bar{c}) = \phi(G, \bar{d}) = H^g$ for some $g \in G$ $\iff$ for all $k \in G$, $(H^g)^k = H^{gk} = \phi(G, \bar{c}^k) = \phi(G, \bar{d}^k) \iff \bar{c}^k/E = \bar{d}^k/E$.

Claim: $\langle G,\ C/E\rangle \cong \langle G,\ H^G\rangle$ via the map $\beta : \bar{b}^g/E \to H^g$. Clearly, β is surjective. Moreover

$$\begin{aligned} H^g = H^h &\iff \phi(G, \bar{b}^g) = \phi(G, \bar{b}^h) \\ &\iff \bar{b}^g E \bar{b}^h \\ &\iff \bar{b}^g/E = \bar{b}^h/E \end{aligned}$$

so β is well-defined and injective. Also, β is a G-morphism, for

$$\beta((\bar{b}/E)^h) = \beta(\bar{b}^h/E) = H^h = (\beta(\bar{b}/E))^h.$$

We now have a chain of G-isomorphisms

$$\begin{aligned} \langle G,\ C/E\rangle &\cong \langle G,\ H^G\rangle \\ &\cong \langle G,\ \cos\ (G : H)\rangle \text{ by 1.2.8} \\ &\cong \langle G,\ \mathcal{M}\rangle \end{aligned}$$

which proves our statement. $\Box$

1.3 A sporadic example

In an unpublished paper, A. Singerman [26] gives a weak $\forall\exists$ interpretation for some important examples of structures which do not have the small index property. These are variations on certain constructions due to Hrushovski: let L be the language containing a $2n$-ary relation symbol E_n for all $n \in \omega$, $n \neq 0$. Consider the class κ of finite L- structures $\mathcal{A}$, where each E_n is interpreted as an equivalence relation on the collection of subsets of $\mathcal{A}$ of size n with at most n equivalence classes. This is an amalgamation class, so it has a Fraïssé limit $\mathcal{M}$ which is ω-categorical. Hrushovski proves that the small index property does not hold for a structure similar to $\mathcal{M}$, and the proof extends to Singerman's construction. In [26], a weak $\forall\exists$ interpretation for $\mathcal{M}$ is produced. A further example handled by Singerman is the following: consider a second language L' containing an n-ary relation symbol R_n for all $n \in \omega$. Let κ' be the class of finite L-structures $\mathcal{B}$ where R_n is interpreted as follows: for each n-tuple $b_1, \ldots, b_n$ of distinct elements from $\mathcal{B}$ there is exactly one permutation of the b_i satisfying R_n. Then the Fraïssé limit $\mathcal{N}$ of κ' has a weak $\forall\exists$ interpretation.

On the other hand, the example $\mathcal{M} = \langle \Omega, B \rangle$ given in Section 1.2 shows that there are structures for which no weak $\forall\exists$ interpretation can be found, yet the small index property holds[2]. Singerman's proof and our result show that having the small index property and having a weak $\forall\exists$ interpretation are two independent conditions.

The small index property is not known to hold for very familiar examples like the countable universal homogeneous partial order, or the countable homogeneous universal tournament. These are in fact handled by Rubin's method: the universal poset appears among the "sporadic examples" in [25], and the tournament is included in the class of *simple structures*[3]. Below we give a weak $\forall\exists$ interpreta-

[2]see also Chapter 3 below for an important class of structures where this happens.

[3]the notion of simplicity in this context does **not** coincide with the terminology used in stability/simplicity theory.

tion for another kind of structure where small index has not been proved, and we prove an easy generalisation of Rubin's result about simple structures.

1.3.1 A doubly ordered structure

We shall here give a weak $\forall\exists$ interpretation for a countable set with two independent dense linear orders without endpoints. Current methods for the small index property (piecewise patching of partial isomorphisms and ample generic automorphisms) seem not to be applicable to this example, which does not appear in Rubin's paper, either. The interpretation we give is based on Rubin's $\forall\exists$ formula for the countable homogeneous universal partial order ([25], pp. 240–243).

We start by building our structure $\mathcal{Q}$.

Proposition 1.3.1 *There is a countable homogeneous structure $\mathcal{Q} = \langle \mathbb{Q}, <_1, <_2 \rangle$, unique up to isomorphism, such that each of $<_1$ and $<_2$ is a dense linear order without endpoints, and every finite structure totally ordered by $<_1$ and $<_2$ embeds in $\mathcal{Q}$.*

Proof $\mathcal{Q}$ is obtained as the Fraissé limit of the class C of finite structures having two independent linear orders. We sketch a proof of the amalgamation property: let $\mathcal{A}$ and $\mathcal{B}$ be finite and such that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}, \mathcal{B}$. Then there is a structure $\mathcal{C} \in \mathrm{C}$ such that $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{C}$, where each order $<_i$ is amalgamated independently as follows (see [20], p. 46, 2.2.1):

i) if $c_1, c_2 \in \mathcal{A}$ (resp. $\mathcal{B}$) and $c_1 <_i c_2$ then put $c_1 <_i c_2$ in $\mathcal{C}$;

ii) if $c_1 \in \mathcal{A} \setminus \mathcal{B}$ and $c_2 \in \mathcal{B} \setminus \mathcal{A}$ and there is no a in $A \cap B$ such that $c_1 <_i a$ and $a <_i c_2$ then put $c_2 <_i c_1$ in $\mathcal{C}$.

iii) if $c_1 \in \mathcal{A} \setminus \mathcal{B}$ and $c_2 \in \mathcal{B} \setminus \mathcal{A}$ and there is $a \in A \cap B$ such that $c_1 <_i a$ in $\mathcal{A}$ and $c_2 >_i a$ in $\mathcal{B}$, then put $c_1 <_i c_2$ in $\mathcal{C}$. □

The ω-categoricity of $\mathcal{Q}$ follows from its construction as a Fraïssé limit. However, in the next proposition we give a proof via a back and forth argument. This requires an explicit axiomatisation of $\mathcal{Q}$, where the axioms express that each $<_i$ is a dense linear order without endpoints, and that all consistent 1-point extensions over any 4 points are realised.

Proposition 1.3.2 *$\mathcal{Q}$ satisfies the following and it is unique among countable structures up to isomorphism.*

1i. $\forall x(\neg x <_i x)$ *for* $i = 1, 2$

2i. $\forall xy(x = y \vee x <_i y \vee y <_i x)$ *for* $i = 1, 2$

3i. $\forall xyz(x <_i y \wedge y <_i z \to x <_i z)$ *for* $i = 1, 2$

4i. $\forall xy(x <_i y \to \exists z(x <_i z <_i y))$ *for* $i = 1, 2$

5i. $\forall x\exists y(y <_i x)$ *for* $i = 1, 2$

6i. $\forall x\exists y(x <_i y)$ *for* $i = 1, 2$

7. $\forall xyuw(x <_1 y \wedge u <_2 w \to \exists z(z <_1 x \wedge z <_2 u))$

8. $\forall xyuw(x <_1 y \wedge u <_2 w \to \exists z(z <_1 x \wedge u <_2 z <_2 w))$

9. $\forall xyuw(x <_1 y \wedge u <_2 w \to \exists z(z <_1 x \wedge w <_2 z))$

10. $\forall xyuw(x <_1 y \wedge u <_2 w \to \exists z(x <_1 z <_1 y \wedge z <_2 u))$

11. $\forall xyuw(x <_1 y \wedge u <_2 w \to \exists z(x <_1 z <_1 y \wedge u <_2 z <_2 w))$

12. $\forall xyuw(x <_1 y \wedge u <_2 w \to \exists z(x <_1 z <_1 y \wedge w <_2 z))$

13. $\forall xyuw(x <_1 y \wedge u <_2 w \to \exists z(y <_1 z \wedge z <_2 u))$

14. $\forall xyuw(x <_1 y \wedge u <_2 w \to \exists z(y <_1 z \wedge u <_2 z <_2 w))$

15. $\forall xyuw(x <_1 y \wedge u <_2 w \to \exists z(y <_1 z \wedge w <_2 z))$

Proof We first show that any two countable structures satisfying the axioms are isomorphic, via a back and forth argument.

Let $\langle A, <_1, <_2\rangle$, $\langle B, <_1, <_2\rangle$ be countable doubly ordered structures satisfying 1i—15, and let $\{a_n : n \in \omega\}$, $\{b_n : n \in \omega\}$ be enumerations of A, B respectively. We build inductively partial isomorphisms ϕ_n, $n \in \omega$.

Base step Put $a_0\phi := b_0$.

"Forth" step Suppose ϕ_n, n even, has been defined on a finite set $F \subset A$. Index the elements of F in two ways so that

$$a_{0_1} <_1 \ldots <_1 a_{n_1} \text{ and}$$

$$a_{0_2} <_2 \ldots <_2 a_{n_2}.$$

Let $b_{i_k} := a_{i_k}\phi_n$.

Pick the least j such that $a_j \notin F$. Then the following cases may occur:

1. $a_j <_1 a_{0_1}$, $a_j <_2 a_{0_2}$. By axiom 7. there is $b_l \in B$ such that $b_l <_1 b_{0_1}(<_1 b_{1_1})$, $b_j <_2 b_{0_2}(<_2 b_{1_2})$. Choose l to be the least index for which this happens and set $\phi_{n+1} := \phi_n \cup \{\langle a_j, b_k\rangle\}$.

2. $a_j <_1 a_{0_1}$, $a_{r_2} <_2 a_j <_2 a_{(r+1)_2}$ for some $r \in \{0, \ldots, n-1\}$. We need to find $b_l \in B$ such that $b_l <_1 b_{0_1}$, $b_h <_2 b_l <_2 b_{(h+1)_2}$. Such a b_l exists by axiom 8. Choose l to be least and set $\phi_{n+1} := \phi_n \cup \{\langle a_j, b_k\rangle\}$.

3. $a_j <_1 a_{0_1}$, $a_{n_2} <_2 a_j$. This and the following cases are treated similarly to 1. and 2. above, by using axioms 9.—15. respectively.

4. $a_{h_1} <_1 a_j <_1 a_{(h+1)_1}$, $a_j <_2 a_{0_2}$

5. $a_{h_1} <_1 a_j <_1 a_{(h+1)_1}$, $a_{r_2} <_2 a_j <_2 a_{(r+1)_2}$ for some $h, r \in \{0, \ldots, n-1\}$

6. $a_{h_1} <_1 a_j <_1 a_{(h+1)_1}$, $a_{n_2} <_2 a_j$

7. $a_{n_1} <_1 a_j$, $a_j <_2 a_{0_2}$

8. $a_{n_1} <_1 a_j$, $a_{r_2} <_2 a_j <_2 a_{(r+1)_2}$

9. $a_{n_1} <_1 a_j$, $a_{n_2} <_2 a_j$

"Back" step Let $G = \mathrm{ran}(\phi_n)$, n odd. Let j be the least integer such that $b_j \notin G$. Then as in the "forth" step we find the least integer k such that a_k is in the same relative orders with respect to $(G)\phi_n^{-1}$ as b_j is with respect to G, and we put $\phi_{n+1} := \phi_n \cup \{\langle a_k, b_j \rangle\}$.

It now suffices to show that $\mathcal{Q}$ satisfies the axioms. This is a straightforward consequence of the universality of $\mathcal{Q}$. We show axiom 11. holds. Suppose $a, b, c, d \in \mathcal{Q}$ are such that $a <_1 b$ and $c <_2 d$. Let M be the 5-element double order $m_1 <_1 m_2 <_1 m_3$, $m_4 <_2 m_2 <_2 m_5$. By universality of $\mathcal{Q}$, there is an embedding $\alpha : M \to \mathcal{Q}$ with $(m_1, m_2, m_3, m_4)\alpha = (a, b, c, d)$. Then $a <_1 m_2\alpha <_1 b$, $c <_2 m_2\alpha <_2 d$, so $m_2\alpha$ is the required witness. The other axioms are treated similarly. $\square$

1.3.2 The interpretation

Let ϕ_0 be the following formula in the language of groups:

$$\phi_0(g, x, y) \equiv x \sim y \sim g \wedge xy \sim g$$

where $\sim$ denotes conjugacy. Then ϕ_0 is an existential formula, for

$$\phi_0(g, x, y) \equiv \exists uvz\ (x^u = g \wedge y^v = g \wedge (xy)^z = g).$$

We show $\mathcal{Q}$ has a weak $\forall\exists$ interpretation based on the following formula ϕ:

$$\phi_0(g, x, y) \wedge (\phi_0(g, x, y) \text{ defines a conjugacy invariant equivalence relation})$$

where $g \in \mathrm{Aut}(\mathcal{Q})$ is an automorphism having a unique fixed point and two infinite orbitals. By Lemma 1.1.3, ϕ is indeed an $\forall\exists$ equivalence formula. We give an explicit construction of the automorphism g which we need in order to define our conjugacy class.

Definition 1.3.3 *A countable set $A = \{a_j : j \in \mathbf{Z}\} \subseteq \mathcal{Q}$ is said to be a* ***cofinal Z-chain*** *in B with respect to $<_i$ if:*

i) *$a_z <_i a_{z+1}$ for all $z \in \mathbf{Z}$;*

ii) *for each $q \in B \setminus A$ there is z such that $a_z <_i q <_i a_{z+1}$ for $i + 1, 2$.*

Given $g \in \mathrm{Aut}(\mathcal{Q})$ and $q \in \mathcal{Q}$ we shall write $\mathrm{Orb}(q, g)$ for the set $\{q^{g^z} : z \in \mathbf{Z}\}$.

Theorem 1.3.4 *There is $g \in \mathrm{Aut}(\mathcal{Q})$ such that:*

1. *g has a single fixed point a_g;*

2. *for $i = 1, 2$: $\forall q \in \mathcal{Q}\ (q \neq a_g \to q <_i q^g)$;*

3. *there is $q \in \mathcal{Q}$ such that for each $i = 1, 2$ $\mathrm{Orb}(q, g)$ is a cofinal $\mathbf{Z}$-chain in $\{x \in \mathcal{Q} : x >_i a_g\}$ with respect to $<_i$ for $i = 1, 2$;*

4. *there is $q' \in \mathcal{Q}$ such that $\mathrm{Orb}(q', g)$ is a cofinal $\mathbf{Z}$-chain in $\{x \in \mathcal{Q} : a_g >_i x\}$ with respect to $<_i$ for $i = 1, 2$.*

Proof 1. Choose some $a \in \mathcal{Q}$. We build sets $A = \{a_n : n \in \omega\}$, cofinal in both $\{x : x >_1 a\}$ and $\{x : x >_2 a\}$, and $B = \{b_n : n \in \omega\}$, cofinal in both $\{x : x <_1 a\}$ and $\{x : x <_2 a\}$; a will eventually become our fixed point.

Enumerate $\mathcal{Q} = \{q_n : n \in \omega\}$.

Base case By universality of $\mathcal{Q}$ find a_0, b_0 such that $a_0 >_i a$ and $b_0 <_i a$ for $i = 1, 2$ (this is a consistent configuration). Put $A_0 = \{a_0\}$, $B_0 = \{b_0\}$.

Inductive step Suppose you have built A_n and B_n. For any set $X \subseteq \mathcal{Q}$ and $q \in \mathcal{Q}$, we shall write $q >_i X$ if $q >_i x$ for all $x \in X$. Pick $q_n \in \mathcal{Q}$. Put $A_{n+1} := A_n \cup \{a_{n+1}\}$ and $B_{n+1} := B_n \cup \{b_{n+1}\}$ with a_{n+1}, b_{n+1} chosen so that, for each $i = 1, 2$:

1. if $a <_i q_n <_i A_n$, then $a <_i a_{n+1} <_i q_n$;

2. if $A_n <_i q_n$, then $q_n <_i a_{n+1}$;

3. if $B_n <_i q_n <_i a$, then $q_n <_i b_{n+1} <_i a$;

4. if $q_n <_i B_n$, then $b_{n+1} <_i q_n$.

Now let $A := \bigcup_{n\in\omega} A_n$, $B := \bigcup_{n\in\omega} B_n$. Clearly, each of the cases above will occur countably many times, thus ensuring that A has no sup and no min, although it has a as an inf, and B has no inf and no max, and has a as a sup. So we can re-index the elements of A and B so that $a_z <_i a_{z+1}$ and $b_z <_i b_{z+1}$ for $z \in \mathbf{Z}$ and define a partial automorphism g_0 of $\mathcal{Q}$ by $a^{g_0} := a$, $a_z^{g_0} := a_{z+1}$, $b_z^{g_0} := b_{z+1}$.

2. $A \cup B$ has the following property:

$$\forall q \in \mathcal{Q} \text{ there is a finite } C \subset A \cup B \cup \{a\} \text{ such that } \forall p \in \mathbb{Q}$$

$$\mathrm{tp}(p/C) = \mathrm{tp}(q/C) \Rightarrow \mathrm{tp}(p/A \cup B \cup \{a\}) = \mathrm{tp}(q/A \cup B \cup \{a\}). \qquad (*)$$

In other words, the type of q over $A \cup B$ is determined by a finite subset: for $q \in \mathcal{Q}$ choose d^1, c^1 resp. least and greatest in $A \cup B$ such that $c^1 <_1 q <_1 d^1$, and d^2, c^2 resp. least and greatest in $A \cup B$ such that $c^2 <_2 q <_2 d^2$. Set $C := \{c^1, d^1, c^2, d^2, a\}$. Then it can be checked that in all cases $\mathrm{tp}(q/C)$ determines $\mathrm{tp}(q/A \cup B \cup \{a\})$.

3. We now define inductively partial automorphisms g_n extending g_0 and such that $\mathrm{Dom}(g_n) \setminus \mathrm{Dom}(g_0)$ is finite.
Enumerate $\mathcal{Q} \setminus \mathrm{Dom}(g_0)$. At stage n, pick $q_n \in \mathcal{Q} \setminus \mathrm{Dom}(g_0)$ and let $C = \{c^1, d^1, c^2, d^2\}$ determine $\mathrm{tp}(q_n, \mathrm{Dom}(g_0))$ by property $(*)$ above. Now $\mathrm{Dom}(g_n) = \mathrm{Dom}(g_0) \cup F$, where F is a finite set. Therefore $\mathrm{tp}(q_n, \mathrm{Dom}(g_n))$ is determined by $\mathrm{tp}(q_n, C \cup F)$. By homogeneity, there is $\tilde{g} \in \mathrm{Aut}(\mathcal{Q})$ agreeing with g_0 on $C \cup \{q_n\}$. We extend g_n to g_{n+1} by putting $q_n^{g_{n+1}} := q_n^{\tilde{g}}$. Note that $q_n^{g_{n+1}} > q_n$.

Eventually we get $g := \bigcup_{n\in\omega} g_n$ as required. $\square$

Definition 1.3.5 *Let $g \in \mathcal{Q}$ satisfy properties 1.— 4. in 1.3.4. Then g is said to be a* ***good*** *automorphism.*

Proposition 1.3.6 *If $g, h \in \mathrm{Aut}(\mathcal{Q})$ are good, then $g \sim h$.*

Proof Following Lemma 4.6 (b) in [25], let g, h have fixed points a_g, a_h respectively. We shall build a sequence of partial isomorphisms k_n between $\langle \mathcal{Q}, g\rangle$ and $\langle \mathcal{Q}, h\rangle$.

Enumerate $\mathcal{Q} = \{q_n : n \in \omega\}$ and set $k_0 = \{\langle a_g, a_h\rangle\}$.
Suppose k_n has been built. Pick $q_n \in \mathcal{Q}$.
If $q_n \in \mathrm{Dom}(k_n)$ then $k_{n+1} := k_n$.
Otherwise let $C \subset \mathrm{Dom}(k_n)$ be a finite set determining $tp(q_n, \mathrm{Dom}(k_n))$ in the sense of 1.3.4 2., that is:

$$\forall p \in \mathcal{Q},\ \mathrm{tp}(q_n, C) = \mathrm{tp}(p, C) \text{ implies } \mathrm{tp}(q_n, \mathrm{Dom}(k_n)) = \mathrm{tp}(p, \mathrm{Dom}(k_n))$$

(it is immediate that such a C exists).
By homogeneity, we can find a corresponding $p_n \in \mathcal{Q}$ such that $\mathrm{tp}(q_n, C) = \mathrm{tp}(p_n, k_n(C))$. Therefore

$$\mathrm{tp}(q_n, \mathrm{Dom}(k_n)) = \mathrm{tp}(p_n, \mathrm{Ran}(k_n))$$

and we can define $k_{n+1} := k_n \cup \{\langle q_n^{g^z}, p_n^{h^z}\rangle : z \in \mathbf{Z}\}$.

Eventually we get $k := \bigcup_{n\in\omega} k_n$, an isomorphism between $\langle \mathcal{Q}, g\rangle$ and $\langle \mathcal{Q}, h\rangle$. $\square$

Proposition 1.3.7 *Let $g, h \in \mathrm{Aut}(\mathcal{Q})$ be good. Then gh is good if and only if $a_g = a_h$.*

Proof Suppose $a_g = a_h$. Then gh has a unique fixed point $a_{gh} = a_g = a_h$ and it is easily proved that properties 1.3.4 1. to 4. are preserved under composition.

Conversely, suppose $a_g \neq a_h$. Then gh moves both a_g and a_h and has no other fixed points (being increasing). Hence it is not good. $\square$

This proves that $\mathcal{Q}$ has a weak $\forall\exists$ interpretation as specified at the beginning of Section 1.3.2.

1.4 Fusion of primitive simple structures

In this section we show that the property of *simplicity*, defined by Rubin for ω-categorical homogeneous structures without algebraicity, is preserved under a certain construction, 'fusion', which amounts to superimposing a structure on another living on the same domain. Rubin shows that simple structures have a weak $\forall\exists$ interpretation. Therefore, fusion of primitive simple structures yields new examples of $\forall\exists$ interpretations. We refer the reader to [25], section 3, p. 234, for the definition of simplicity in its full generality. Here we give the definition in the special case of a primitive structure. We adopt Rubin's notation and terminology throughout the section.

Definition 1.4.1 (Primitive simple structure) *Let $\mathcal{M}$ be a primitive homogeneous L-structure without algebraicity, where L is a relational language containing binary predicates only. Let S be a 2-type of $\mathcal{M}$. Then $\mathcal{M}$ is said to be S-**simple** if for every finite subset $A \subseteq \mathcal{M}$ and $b, c \in \mathcal{M} \setminus A$, there is $c' \in \mathcal{M}$ such that*

1. $\mathrm{tp}(c'/A) = \mathrm{tp}(c/A)$;
2. $\mathrm{tp}(bc') = S$.

The following proposition formalises the fusion construction mentioned above. We state the construction for two general structures without algebraicity, although we shall only use the case where $\mathcal{M}_1$ and $\mathcal{M}_2$ are primitive.

Proposition 1.4.2 *Let $\mathcal{M}_1, \mathcal{M}_2$ be two homogeneous structures without algebraicity in disjoint relational binary languages L_1, L_2 respectively. Then there exists a homogeneous $(L_1 \cup L_2)$-structure $\mathcal{M}$ such that $\mathcal{M}|_{L_i} \cong \mathcal{M}_i$ for $i = 1, 2$, and such that $\mathcal{M}$ is universal for finite structures whose reduct to L_i embeds in $\mathcal{M}_i$. Moreover, $\mathcal{M}$ has no algebraicity.*

Proof $\mathcal{M}$ is obtained as the Fraissé limit of the class of finite structures whose reduct to L_i embeds in $\mathcal{M}_i$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ have no algebraicity, by Proposition 3.9 in [6], neither does $\mathcal{M}$. $\square$

Lemma 1.4.3 *Suppose $\mathcal{M}$ is a primitive S-simple structure, and let a, $b \in \mathcal{M}$. Then for any $k > 1$ we can find in $\mathcal{M}_i$ an S-path of length k from a to b, i.e. a sequence $a = a_0, a_1, .., a_k = b$ with $S(a_j, a_{j+1})$ or $S(a_{j+1}, a_j)$.*

Proof It suffices to prove that for any two points c, $d \in \mathcal{M}$ there is an S-path of length 2 from c to d. Then we find inductively a path of length $k+1$ between a and b by adjoining to a path $a = a_0, a_1, .., a_k = b$ a path of length 2 between a_{k-1} and b.

For $c \in \mathcal{M}$ there is $c' \neq d$ such that $S(c, c')$. By the definition of an S-simple structure, there is $c'' \in \mathcal{M}$ such that $\mathrm{tp}(c, c'') = \mathrm{tp}(c, c') = S$ and $S(d, c'')$. The c, c'', d is the required path. $\square$

Proposition 1.4.4 *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be primitive structures in two disjoint binary languages L_1 and L_2 respectively. Suppose that $\mathcal{M}_i$ is S_i-simple, for $i = 1, 2$. Let $\mathcal{M}$ be their fusion, as in proposition 1.4.2. Then $\mathcal{M}$ is primitive.*

Proof Let R be a 2-type of $\mathcal{M}$. Then R is of form $R_1 \wedge R_2$, R_i a 2-type of $\mathcal{M}_i$. Consider the graph on $\mathcal{M}$ where x is joined to y if Rxy or Ryx. By D. G. Higman's criterion for primitivity (cf. [4], Theorem 5.7), it suffices to show that this graph is connected. We know the corresponding graphs $(\mathcal{M}_i, R_i)$ are connected. We also have the 'special' 2-types S_i of $\mathcal{M}_i$.

Pick $a, b \in M$. We must find an R-path from a to b. By 1.4.3, for any $k > 1$ we can find in $\mathcal{M}_i$ an S_i-path of length k from a to b, i.e. a sequence $a = a_0, a_1, .., a_k = b$ with $S_i(a_j, a_{j+1})$ or $S_i(a_{j+1}, a_j)$.

We claim that for each $i = 1, 2$ there is $m_i > 0$ such that if $c, d \in \mathcal{M}_i$ are S_i-joined then there is an R_i-path of length m_i from c to d. To see this, suppose $S_i(c_0, d_0)$. By primitivity of $\mathcal{M}_i$, there is an R_i-path, of length m_i say, from c_0

to d_0. There is $g \in \mathrm{Aut}(\mathcal{M}_i)$ such that $g(c_0) = c$, $g(d_0) = d$. Then g takes the R_i-path from c_0 to d_0 to one from c to d.

Thus, for any $k > 0$ there is an R_i-path of length km_i from a to b, in the structure $\mathcal{M}_i$. In particular, there is an R_1 path of length m_1m_2 from a to b in $\mathcal{M}_1$, and an R_2 path of length m_1m_2 from a to b in $\mathcal{M}_2$. There is an $(L_1 \cup L_2)$-structure consisting of a, b with $R_1(a,b) \wedge R_2(a,b)$ and a set of $m_1m_2 - 2$ points which forms both an R_1- and an R_2-path. By universality of $\mathcal{M}$ there is a copy of this structure in $\mathcal{M}$, and by homogeneity it can be chosen to be over $\{a, b\}$. So we have found an $R_1 \wedge R_2$ path from a to b, of length m_1m_2, in $\mathcal{M}$. $\Box$

Proposition 1.4.5 *If $\mathcal{M}_1$, $\mathcal{M}_2$ are primitive structures in disjoint binary languages such that $\mathcal{M}_i$ is S_i-simple for $i = 1, 2$. Then their fusion $\mathcal{M}$ is primitive and $S_1 \wedge S_2$-simple.*

Proof By 1.4.2, $\mathcal{M}$ is homogeneous and has no algebraicity. By 1.4.4, it is primitive. Suppose that $\mathcal{M}_i$ is S_i-simple for $i = 1, 2$, and let $S = S_1 \wedge S_2$. Let $A \subseteq \mathcal{M}$ be a finite subset, and let $b, c \in \mathcal{M} \setminus A$ be such that $b \neq c$. Then, for $i = 1, 2$: by simplicity of $\mathcal{M}_i$ there is c_i' such that

1. $\mathrm{tp}_{\mathcal{M}_i}(c_i/A) = \mathrm{tp}_{\mathcal{M}_i}(c_i'/A)$;
2. $\mathrm{tp}_{\mathcal{M}_i}(bc_i') = S_i$.

By universality and homogeneity of $\mathcal{M}$ we can choose $c_1' = c_2' = c'$. Then $\mathrm{tp}_{\mathcal{M}}(bc') = S_1 \wedge S_2$, as required. $\Box$

1.5 Weak $\forall\exists$ interpretations on normal subgroups

We now give a further method for obtaining weak $\forall\exists$ interpretations from existing ones: consider a normal subgroup H of the full automorphism group of an $\aleph_0$-categorical structure $\mathcal{M}$. If $\langle H, \mathcal{M}\rangle$ has a weak $\forall\exists$ interpretation, under certain reasonable conditions on H and on the interpretation, we obtain a weak

$\forall\exists$ interpretation for $\langle \mathrm{Aut}(\mathcal{M}), (\mathcal{M})\rangle$. Below we give the main result, together with a method for proving the necessary definability condition on H, and an easy application.

Proposition 1.5.1 *Let $G = \mathrm{Aut}(\mathcal{M})$, $\mathcal{M}$ an ω-categorical structure, and let $H \lhd G$ be a closed subgroup which is oligomorphic and transitive on $\mathcal{M}$ and $\exists$ definable in G. Suppose $\langle H, \mathcal{M}\rangle$ has a weak $\forall\exists$ interpretation $\langle H, C/E\rangle$ where*

1. *$C \subseteq H^n$ consists of n-tuples of automorphisms $\langle g_0, \ldots, g_n\rangle$ having the same fixed space, that is, $\mathrm{fix}(g_0) = \mathrm{fix}(g_1) = \ldots = \mathrm{fix}(g_n)$;*
2. *the equivalence relation E on C is defined by an existential formula $\phi(x, y, \bar{h})$;*
3. *$\bar{g}E\bar{k}$ if and only if $\mathrm{fix}(g_i) = \mathrm{fix}(k_i)$ for $i = 0, \ldots, n$;*
4. *the bijection τ takes $\langle g_0, \ldots, g_n\rangle/E$ to $m \in \mathrm{fix}(g_0)$.*

Then $\langle G, \mathcal{M}\rangle$ has a weak $\forall\exists$ interpretation.

Proof For ease of notation, we shall take $n = 2$. Let $C = \langle h_0, h_1\rangle^H$ be the conjugacy class involved and ϕ be the existential formula defining the equivalence relation E on C, so that

$$\tau : \langle H, \langle h_0, h_1\rangle^H/E\rangle \cong \langle H, \mathcal{M}\rangle.$$

Let $\hat{C} = \langle h_0, h_1\rangle^G$. By normality of H, $\hat{C} \subseteq H \times H$. We would like to define $\hat{E}$ on $\hat{C}$ so that there is an isomorphism

$$\hat{\tau} : \langle G, \hat{C}/\hat{E}\rangle \cong \langle G, \mathcal{M}\rangle.$$

The obvious choice is to identify elements of $\hat{C}$ which have the same fixed space in $\mathcal{M}$. We know by hypothesis that elements of C, hence of $\hat{C}$, have the same fixed points in their action on $\mathcal{M}$, hence the same happens in their action on C/E. So we can define $\hat{E}$ by identifying $\langle g_0, g_1\rangle, \langle k_0, k_1\rangle \in \hat{C}$ whenever their fixed points in the action on C/E are the same, that is

$$\langle g_0, g_1\rangle \hat{E} \langle k_0, k_1\rangle \quad \text{iff} \quad \forall \langle x_0, x_1\rangle \in C(\ (\langle x_0, x_1\rangle^{g_0} E \langle x_0, x_1\rangle \wedge \langle x_0, x_1\rangle^{g_1} E \langle x_0, x_1\rangle)$$

$$\leftrightarrow (\langle x_0, x_1\rangle^{k_0} E\langle x_0, x_1\rangle \wedge \langle x_0, x_1\rangle^{k_1} E\langle x_0, x_1\rangle)).$$

Note that if $\langle g_0, g_1\rangle, \langle k_0, k_1\rangle \in \hat{C}$, then $\langle x_0, x_1\rangle^{g_i}$ and $\langle x_0, x_1\rangle^{k_i}$ are in C, so $\hat{E}$ is defined. By its form, $\hat{E}$ is an equivalence relation in any group, and it is G-invariant. We claim further that $\hat{E}$ is $\forall\exists$ definable in G in the language of groups.

Let ψ be an $\exists$ formula defining H. Then C is also $\exists$ definable (with parameters h_0, h_1) via the formula

$$\chi(x_0, x_1, h_0, h_1) \equiv \exists y(\psi(y) \wedge \langle x_0, x_1\rangle = \langle h_0, h_1\rangle^y).$$

Let $\bar{x} = \langle x_0, x_1\rangle$. Now it is easy to define $\hat{E}$ by

$$\hat{\phi}(\bar{x}, \bar{y}, h_0, h_1) \equiv \forall \bar{z}(\chi(\bar{z}) \to ((\bar{z}^{x_0} E\bar{z} \wedge \bar{z}^{x_1} E\bar{z}) \leftrightarrow (\bar{z}^{y_0} E\bar{z} \wedge \bar{z}^{y_1} E\bar{z})).$$

The fact that E is $\exists$ definable guarantees that $\hat{E}$ is $\forall\exists$ definable. $\square$

When choosing H to be $\exists$ definable in the hypotheses of the previous proposition, we have in mind the case when H contains a **generic automorphism**, that is, an automorphism which lies in a comeagre conjugacy class (see Definition 2.1.2). If so, the following definability result holds.

Lemma 1.5.2 *Let G be a Polish group, and $H \lhd G$ be a closed normal subgroup which contains an element h generic in H. Then H is $\exists$ definable in G.*

Proof Let $C_H = h^H$ be the comeagre conjugacy class of h. First, as is well known, any element $k \in H$ can be written as the product of two generics. Indeed, C comeagre implies $C \cap kC \neq \emptyset$. Then choose $g_0 \in C \cap kC$, so that $g_0 = kg_1$ for some $g_1 \in C$. Then $k = g_0 g_1^{-1}$. Since $C = C^{-1}$, $g_0, g_1^{-1} \in C$ as required.

Consider now the conjugacy class C_G of h in G. Since $C_H \subseteq C_G$ every element of H is a product of two elements of C_G, and C_G is $\exists$ definable in G with parameter h. So we have $H \subseteq C_G C_G$. By normality of H, $C_G \subseteq H$ so we can define $H = C_G C_G$. $\square$

Example 1.5.3 Consider the countable dense linear order without endpoints $\langle \mathbb{Q}, < \rangle$, and the **linear betweeness relation** B on $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ defined by

$$B(x, y, z) \iff (y < x < z) \vee (z < x < y).$$

Let $\mathcal{M}$ be the structure $\langle \mathbb{Q}, B \rangle$. It is easy to see that:

- $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle) \lhd \mathrm{Aut}(\langle \mathbb{Q}, B \rangle)$;
- $|\mathrm{Aut}(\langle \mathbb{Q}, B \rangle) : \mathrm{Aut}(\langle \mathbb{Q}, < \rangle)| = 2$.

A proof can be found in [4], 11.3.2.

In [29], Truss proves that $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ has a generic automorphism. By 1.5.2, $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ is $\exists$ definable in $\mathrm{Aut}(\langle \mathbb{Q}, B \rangle)$. In [25], Rubin gives a weak $\forall\exists$ interpretation for $\langle \mathbb{Q}, < \rangle$ which satisfies the hypotheses of Proposition 1.5.1. It follows that there is a weak $\forall\exists$ interpretation for $\mathrm{Aut}(\langle \mathbb{Q}, B \rangle)$ acting on $\langle \mathbb{Q}, B \rangle$.

Chapter 2

Relational structures and Baire category

This chapter shows how to obtain a weak $\forall\exists$ interpretation for a homogeneous transitive structure $\mathcal{M}$ whose automorphism group contains a *generic pair* of automorphisms (in a sense modified from the usual one). The method applies to a range of relational structures, including k-kypergraphs, K_m-free graphs and the Henson digraphs, for which the small index property holds, through the proof in [17] and various extension lemmas for partial isomorphisms proved by Herwig [16]. These lemmas are an analogue of Hrushovski's extension lemma for graphs [18], which is needed for the argument in [17] to work for the random graph. Indeed, we need suitable versions of these extension lemmas, relativised to partial isomorphisms having a specific cycle type. The first section of this chapter contains the theory of generic pairs and a description of some sufficient conditions for their existence. In the second section we show how to base a weak $\forall\exists$ interpretation on the existence of such a pair. The third section contains slight modifications of Herwig's arguments in [16], needed for the construction in Section 2.1 to work.

All the examples handled in this chapter are known to have the small index property. However, it is plausible that the method that we give here for obtaining weak $\forall\exists$ interpretations might work where Herwig's method for small index does not.

2.1 Structures with a generic pair of automorphisms

Let $\mathcal{M}$ be a transitive ω-categorical structure, $G = \mathrm{Aut}(\mathcal{M})$, $d \in \mathcal{M}$ and $X_d \subseteq G$ be the set of automorphisms fixing only d:

$$X_d := \{p \in G : \mathrm{fix}(p) = \{d\}\}.$$

It is well known that G is a Polish space (i.e. a completely metrisable space which is also separable).

Fact 2.1.1 *The set X_d is closed in G.*

Proof Suppose $g \in G$ is such that for all $k \in \omega$, $\bar{m} \in \mathcal{M}^k$ there is $h \in X_d$ with $\bar{m}^h = \bar{m}^g$. Then in particular there is $h \in X_d$ such that $d^g = d^h = d$, so g fixes d. Similarly we can conclude that $m^g \neq m$ for all $m \in \mathcal{M}$, $m \neq d$. So g is in fact in X_d, which proves that X_d is closed. $\square$

Since a closed subspace of a Polish space is Polish, we have that X_d and the stabiliser G_d are Polish spaces in their own right. Also, the product of two Polish spaces is again a Polish space ([21], Proposition 3.3), so $X_d \times X_d$ is also Polish.

Definition 2.1.2 *Let X be a topological space. Then:*

- *a set $U \subseteq X$ is said to be* ***comeagre*** *if there are $\{U_i\}_{i\in\omega}$ with each U_i dense and open and*
$$\bigcap_i U_i \subseteq U;$$
- *a set $V \subseteq X$ is* ***meagre*** *if and only if $X \setminus V$ is comeagre;*
- *X is a* ***Baire space*** *if every comeagre set in X is dense;*

- *$A \subseteq X$ has the **Baire property** if there is an open set U such that the symmetric difference $A \triangle U$ is meagre in X.*

Clearly, open sets and meagre sets have the Baire property.

It is well known that every completely metrisable space is a Baire space. By the preceding remarks, X_d and $X_d \times X_d$ are complete metric spaces, so Baire category arguments apply to both these spaces.

The following lemma is a trivial consequence of the definition of a comeagre set:

Lemma 2.1.3 *In a Baire space any two comeagre sets have non-empty intersection.*

Proof Let $U \supseteq \bigcap_i U_i$ and $V \supseteq \bigcap_j V_j$ be comeagre. Then $U \cap V \supseteq (\bigcap_i U_i) \cap (\bigcap_j V_j)$ is comeagre, hence dense, therefore non-empty. $\square$

Definition 2.1.4 *Let $X \subseteq \mathrm{Aut}(\mathcal{M})$ be closed in $\mathrm{Aut}(\mathcal{M})$, so that X is a Polish space with the inherited topology. Suppose $H \leq \mathrm{Aut}(\mathcal{M})$ is a subgroup such that $X^H \subseteq X$, so that H acts on X by conjugation. A tuple $(g_1, \ldots, g_n)$ is an **H-generic tuple** in X if the orbit $(g_1, \ldots, g_n)^H$ of H on X^n is comeagre in the Polish space X^n.*

Fact 2.1.5 *Any two H-generic n-tuples are conjugate in X^n.*

Proof This follows from the fact that orbits of H on X^n are either disjoint or equal, and from 2.1.3. $\square$

We shall be concerned with relational structures whose automorphism group contains a pair (f_1, f_2) of automorphisms such that $\mathrm{fix}(f_1) = \mathrm{fix}(f_2) = \{d\}$ and the pair (f_1, f_2) is generic in $X_d \times X_d$ (i.e. $(f_1, f_2)^{G_d}$ is comeagre in $X_d \times X_d$).

Suppose that $\mathcal{M}$ is an ω-categorical, transitive and homogeneous structure in the relational language $L_0 = \{R_1, \ldots, R_n\}$. Consider the class κ' of all finite substructures of $\mathcal{M}$. For $\mathcal{A} \in \kappa'$, consider an expansion $\mathcal{A}'$ of $\mathcal{A}$ to the language $L = \{R_1, \ldots, R_n, f_1, f_2, d\}$, where f_1 and f_2 are function symbols and d is a constant. Consider the class κ consisting of all those finite $\mathcal{A} \subseteq \mathcal{M}$ where f_1 and f_2 are interpreted as automorphisms of $\mathcal{A}$ such that $\mathrm{fix}(f_1) = \mathrm{fix}(f_2) = \{d\}$. Suppose κ has the amalgamation property, where the amalgamation is "free":

Definition 2.1.6 *Let κ the class of structures described above. Let $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in \kappa$ be such that $\mathcal{A} \subseteq \mathcal{B}_i$, $\mathcal{A} = \mathcal{B}_1 \cap \mathcal{B}_2$ (this can be assumed without loss of generality), and let $f_j^{\mathcal{A}} \subseteq f_j^{\mathcal{B}_i}$, $i, j = 1, 2$. Let $\mathcal{C}$ be the disjoint union of $\mathcal{B}_1$ and $\mathcal{B}_2$ over $\mathcal{A}$ so that:*

1. $\mathcal{B}_i \subseteq \mathcal{C}$, $i = 1, 2$;

2. $\mathcal{C} = \mathcal{B}_1 \cup \mathcal{B}_2$, $f_i^{\mathcal{C}} = f_i^{\mathcal{B}_1} \cup f_i^{\mathcal{B}_2}$;

3. for all relation symbols $R_j \in L$ and n-tuples $\bar{a} \in \mathcal{C}^n$, $\mathcal{C} \models R_j\bar{a}$ if and only if $\bar{a} \in \mathcal{B}_i$ for some i and $\mathcal{B}_i \models R_j\bar{a}$.

*Then $\mathcal{C}$ is called the **free amalgam** of $\mathcal{B}_1$ and $\mathcal{B}_2$.*

*If for all $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in \kappa$ we have $\mathcal{C} \in \kappa$, we say that κ has the **free amalgamation property**.*

Free amalgamation is generally treated as a property of structures in a relational language. As such, k-hypergraphs, K_n-free graphs and, more generally, the class of structures described by Herwig in [16] all enjoy free amalgamation. The property does not hold, for instance, for tournaments or ordered structures. In our case, the presence of two function symbols does not affect free amalgamation (to see this, one can take the two functions as interpreting two binary relation symbols).

So a typical element of κ is a finite structure of the form $(\mathcal{A}, f_1, f_2, d)$. If we assume that structures in κ can be amalgamated freely, Fraïssé's theorem ensures that κ has a Fraïssé limit $(\mathcal{M}, f_1, f_2, d)$, which is countable and homogeneous.

Our claim is that then the automorphisms f_1, f_2 form a generic pair in $X_d \times X_d$. The proof is via a Banach-Mazur game, and it requires the class of structures in question to satisfy a fixed point extension property for finite partial isomorphisms having a single fixed point:

Definition 2.1.7 *Let S be a relational language, π a class of finite S-structures. Then π is said to have the* ***fixed point extension property*** *(FEP) for finite partial isomorphisms if for all $\mathcal{A} \in \pi$, and $p_1, \ldots, p_n$ finite partial isomorphisms of $\mathcal{A}$ such that $\mathrm{fix}(p_1) = \cdots = \mathrm{fix}(p_n) = \{d\}$, there are $\mathcal{B} \in \pi$ such that $\mathcal{A} \subseteq \mathcal{B}$, and $f_1, \ldots, f_n \in \mathrm{Aut}(\mathcal{B})$ with $p_i \subseteq f_i$ and $\mathrm{fix}(f_i) = \{d\}$ for $i = 1, \ldots, n$.*

We shall use the property with $n = 2$, which we call FEP_2. Section 3 below will be devoted to proving FEP for a range of different classes of relational structures.

We check that in our construction of $(\mathcal{M}, f_1, f_2, d)$, the reduct to L_0 is the universal homogeneous structure $\mathcal{M}$ we started off with:

Lemma 2.1.8 *Let $(\mathcal{M}, f_1, f_2, d)$ be the Fraïssé limit of the class κ of finite structures in the language L described above, and suppose κ has FEP_2. Then the reduct $(\mathcal{M}, f_1, f_2, d)|_{L_0}$ is isomorphic to $\mathcal{M}$.*

Proof Let $\mathcal{A}$ be a finite substructure of $\mathcal{M}$. We want to show that for any finite L_0 structure $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{B}$ embeds in $(\mathcal{M}, f_1, f_2, d)|_{L_0}$ over $\mathcal{A}$.

Since $(\mathcal{M}, f_1, f_2, d)$ is the Fraïssé limit of κ, it is a union of a chain of members of κ. Hence there is a finite L-structure $(\mathcal{A}', f_1, f_2, d) \subseteq (\mathcal{M}, f_1, f_2, d)$, $(\mathcal{A}', f_1, f_2, d) \in \kappa$, such that $\mathcal{A} \subseteq (\mathcal{A}', f_1, f_2, d)|_{L_0}$. Let $\mathcal{C}$ be the free L_0-amalgam of $(\mathcal{A}', f_1, f_2, d)|_{L_0}$ and $\mathcal{B}$. Regard $\mathcal{C}$ as an L-structure where the function symbols f_1, f_2 are interpreted as the finite partial isomorphisms induced by $f_1, f_2 \in \mathrm{Aut}(\mathcal{A}')$. By FEP_2 there are a finite L-structure $\mathcal{C}' \in \kappa$ and automorphisms $f_1', f_2' \in \mathrm{Aut}(\mathcal{C}')$ such that:

1. $\mathcal{C} \subseteq \mathcal{C}'$;
2. $f_i \subseteq f_i'$ for $i = 1, 2$;
3. $\mathrm{fix}(f_i') = \{d\}$ for $i = 1, 2$.

By the universality and homogeneity of $(\mathcal{M}, f_1, f_2, d)$ with respect to structures in κ, $\mathcal{C}'$ embeds in $(\mathcal{M}, f_1, f_2, d)$ over $\mathcal{A}'$. It follows that $\mathcal{B}$ embeds in $(\mathcal{M}, f_1, f_2, d)|_{L_0}$ over $\mathcal{A}$, as required. $\square$

We now prove that the automorphisms f_1, f_2 of $(\mathcal{M}, f_1, f_2, d)$ constructed above are a generic pair.

Proposition 2.1.9 *Let $\mathcal{M}$ be an ω-categorical, transitive and homogeneous structure in the relational language $L_0 = \{R_1, \ldots, R_n\}$. Consider the class κ' of all finite substructures of $\mathcal{M}$. For $\mathcal{A} \in \kappa'$, consider an expansion $\mathcal{A}'$ of $\mathcal{A}$ to the language $L = \{R_1, \ldots, R_n, f_1, f_2, d\}$, where f_1 and f_2 are function symbols and d is a constant. Consider the class κ consisting of all those finite $\mathcal{A} \subseteq \mathcal{M}$ where f_1 and f_2 are interpreted as automorphisms of $\mathcal{A}$ such that $\mathrm{fix}(f_1) = \mathrm{fix}(f_2) = \{d\}$. Suppose κ has FEP$_2$, is closed under taking substructures and has the free amalgamation property. Let $(\mathcal{M}, f_1, f_2, d)$ be the Fraïssé limit of κ, let $d = \mathrm{fix}(f_1) = \mathrm{fix}(f_2)$, $G = \mathrm{Aut}(\mathcal{M})$, and let $\mathcal{D} = (f_1, f_2)^{G_d}$. Then $\mathcal{D}$ is comeagre in $X_d \times X_d$.*

Proof We play the Banach-Mazur game of $\mathcal{D}$. Let
$P := \{f : \mathcal{M} \to \mathcal{M} : f$ is a finite partial isomorphism such that $\mathrm{fix}(f) = d\}$.
P is partially ordered by inclusion. Now let $P^2 = P \times P$. The game is played as follows: players I and II choose an increasing sequence of elements of P^2

$$(p_{1,0},\ p_{2,0}), (p_{1,1},\ p_{2,1}), (p_{1,2},\ p_{2,2}), \ldots$$

so that $p_{1,i} \subseteq p_{1,i+1}$ and $p_{2,i} \subseteq p_{2,i+1}$ for all i. Player I starts the game and chooses $(p_{1,i}, p_{2,i})$ for i even, player II chooses at odd stages. Player II wins if and only if $(p_1, p_2) := (\bigcup_{i\in\omega} p_{1,i}, \bigcup_{i\in\omega} p_{2,i}) \in \mathcal{D}$. Player II has a winning strategy iff $\mathcal{D}$ is comeagre in $X_d \times X_d$.

Player II can always play so that at stage i, for $i > 1$ and even,

1. he can choose to put any $x \in \mathcal{M}$ in the domain and range of $p_{1,i}, p_{2,i}$;
2. $(p_{1,i}, p_{2,i}) \in P^2$ and $\mathrm{dom}(p_{1,i}) = \mathrm{ran}(p_{1,i})$, $\mathrm{dom}(p_{2,i}) = \mathrm{ran}(p_{2,i})$;
3. $(\mathcal{M}, p_1, p_2, d)$ is *weakly homogeneous*, that is: if $(\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d)$, $(\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d)$ are finite L-structures, and $(\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d) \subseteq (\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d)$, and $\alpha : (\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d) \to (\mathcal{M}, p_1, p_2, d)$ is an embedding, there is an embedding $\tilde{\alpha} : (\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d) \to (\mathcal{M}, p_1, p_2, d)$ extending α.

At stage $i+1$, i even, player II is given a finite structure $(\Delta_i, p_{1,i}, p_{2,i}, d)$, where the $p_{j,i}$ are finite partial isomorphisms of Δ_i. For points 1. and 2., for any $x \in \mathcal{M}$, II can consider $\Delta'_{i+1} := \Delta_i \cup \{x\}$ and use FEP$_2$ to obtain extensions Δ_{i+1} of Δ'_{i+1}, and $p_{1,i+1}, p_{2,i+1} \in \mathrm{Aut}(\Delta_{i+1})$ of $p_{1,i}, p_{2,i}$, each fixing only d.

In order for 3. to hold, a typical task for II is the following: for $(\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d) \subseteq (\Delta_i, p_{1,i}, p_{2,i}, d)$ and $(\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d) \supseteq (\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d)$, II has to ensure that $(\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d)$ embeds in $(\Delta_{i+1}, p_{1,i+1}, p_{2,i+1}, d)$ over $(\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d)$. So II wants to create an amalgam of $(\Delta_i, p_{1,i}, p_{2,i}, d)$ and $(\mathcal{B}, p_1^{\mathcal{B}}, p_2^{\mathcal{B}}, d)$ over $(\mathcal{A}, p_1^{\mathcal{A}}, p_2^{\mathcal{A}}, d)$, which he can do by free amalgamation. Call this amalgam $(\Delta'_{i+1}, p'_{1,i+1}, p'_{2,i+1}, d)$. By universality of $\mathcal{M}$ as an L_0-structure, we find a copy $(\Delta_{i+1}, p_{1,i+1}, p_{2,i+1}, d)$ of $(\Delta'_{i+1}, p'_{1,i+1}, p'_{2,i+1}, d)$ in $\mathcal{M}$, and by homogeneity we can choose the copy to be over Δ_i, say

$$\begin{aligned} \psi : \ \Delta'_{i+1} &\to \mathcal{M}, \text{ with} \\ \Delta_i &\subseteq \psi(\Delta'_{i+1}) \end{aligned}$$

So II can put $\Delta_{i+1} := \psi(\Delta'_{i+1})$ and $p_{1,i+1} := \psi \circ p'_{1,i+1}$, $p_{2,i+1} := \psi \circ p'_{2,i+1}$.

It follows from 3. that $(\mathcal{M}, p_1, p_2, d)$ is homogeneous and universal for finite L-structures. Hence $(\mathcal{M}, p_1, p_2, d) \cong (\mathcal{M}, f_1, f_2, d)$, so in particular $(p_1, p_2) \sim (f_1, f_2)$. $\square$

Now recall the following facts about comeagre sets:

Theorem 2.1.10 (Kuratowski-Ulam) *Let X, Y be second countable Baire spaces and suppose $A \subseteq X \times Y$ has the Baire property. For $x \in X, y \in Y$ define $A_x := \{y \in Y : (x,y) \in A\}$ and $A^y := \{x \in X : (x,y) \in A\}$. Then the following are equivalent:*

1. *A is comeagre;*

2. *$\{x \in X : A_x$ is comeagre $\}$ is comeagre in X;*

3. *$\{y \in Y : A^y$ is comeagre $\}$ is comeagre in Y.*

Proof [21], 8.41. □

An easy consequence of this theorem is the following:

Fact 2.1.11 *Let X, Y be second countable topological spaces and let $A \subseteq X$, $B \subseteq Y$ be non empty. Then if $A \times B$ is comeagre in $X \times Y$, A is comeagre in X and B is comeagre in Y.*

Proof Since $A \times B$ is comeagre, it has the Baire property. By 2.1.10, there is $x \in A$ such that $(A \times B)_x$ is comeagre in Y. But $(A \times B)_x = B$. □

We can now prove

Lemma 2.1.12 *The set $f_1^{G_d}$ is comeagre in X_d.*

Proof Consider the projections $\mathcal{D}_1$, $\mathcal{D}_2$ of $\mathcal{D}$ to the first and second coordinates respectively. Clearly $\mathcal{D} \subseteq \mathcal{D}_1 \times \mathcal{D}_2$. Since $\mathcal{D}$ is comeagre in $X_d \times X_d$, $\mathcal{D}_1 \times \mathcal{D}_2$ also is. By 2.1.11, $\mathcal{D}_1$ is comeagre in X_d. Note that $f_1^{G_d} = \mathcal{D}_1$. □

We now state our main result:

Proposition 2.1.13 *Let $g \in f_1^{G_d}$ and $\mathcal{D}_g := \{f \in X_d : (g, f) \in \mathcal{D}\}$. Then $\mathcal{D}_g$ is comeagre in X_d for all $g \in f_1^{G_d}$.*

Since $\mathcal{D}$ has the Baire Property, by the Kuratowski-Ulam theorem, the set

$$\{g \in X_d : \mathcal{D}_g \text{ is comeagre in } X_d\}$$

is comeagre in X_d. Also, $f_1^{G_d}$ is comeagre in X_d, hence

$$\{g \in X_d : \mathcal{D}_g \text{ is comeagre in } X_d\} \cap f_1^{G_d} \neq \emptyset.$$

Pick $g \in \{g \in X_d : \mathcal{D}_g \text{ is comeagre in } X_d\} \cap f_1^{G_d}$, so that $\mathcal{D}_g$ is comeagre in X_d. Note that G_d is transitive on $f_1^{G_d}$. Also, if $\mathcal{D}_g$ is comeagre in X_d and h is conjugate to g under G_d, then $\mathcal{D}_h$ is also comeagre in X_d.

Therefore, $\mathcal{D}_g$ is comeagre for all $g \in f_1^{G_d}$. □

2.2 The interpretation

Let $\mathcal{M}$ be an ω-categorical, transitive and homogeneous structure in a relational language which satisfies the hypotheses of Proposition 2.1.9. Then the Fraïssé limit $(\mathcal{M}, f_1, f_2, d)$ constructed in 2.1.9 exists, and $(f_1, f_2) \in \mathrm{Aut}(\mathcal{M})^2$ is a generic pair of automorphisms such that $\mathrm{fix}(f_1) = \mathrm{fix}(f_2) = \{d\}$. We give a weak $\forall\exists$ interpretation for $\mathcal{M}$ based on an equivalence relation defined in terms of our comeagre orbit on pairs $\mathcal{D} = (f_1, f_2)^{G_d}$, with the notation of Section 2.1.

Define $\mathcal{D}^G = \{(g_1, g_2)^g : (g_1, g_2) \in \mathcal{D},\ g \in G\}$, and let $\mathcal{D}_1^G$ be the projection of $\mathcal{D}^G$ to the first coordinate. Note that since we assume G to be transitive, for each $a \in \mathcal{M}$ there is $g \in G$ such that $a^g = d$. The set $\mathcal{D}^G$ consists of certain pairs (h_1, h_2) such that $\mathrm{fix}(h_1) = \mathrm{fix}(h_2)$ is a singleton, and for each $a \in \mathcal{M}$ there is a pair in $\mathcal{D}^G$ fixing a. We shall define an equivalence relation on $\mathcal{D}_1^G$ which identifies automorphisms having the same fixed point.

Lemma 2.2.1 *Let E be the following equivalence relation on $\mathcal{D}_1^G$:*

$$g_1 E g_2 \iff \mathrm{fix}(g_1) = \mathrm{fix}(g_2).$$

Then for $g_1, g_2 \in \mathcal{D}_1^G$

$$g_1\ E\ g_2 \iff \exists f \in G : (g_1, f),\ (g_2, f) \in \mathcal{D}^G,$$

so E is $\exists$-definable with parameters in the language of groups.

Proof $(\Leftarrow)$ is immediate.

$(\Rightarrow)$ Let $g_1, g_2 \in \mathcal{D}_1^G$ have the same fixed point e. Then, by transitivity of G, find a conjugating element $h \in G$ so that $\mathrm{fix}(g_1^h) = \mathrm{fix}(g_2^h) = d$. By 2.1.13, $\mathcal{D}_{g_1^h}$ and $\mathcal{D}_{g_2^h}$ are comeagre in X_d. Hence $\mathcal{D}_{g_1^h} \cap \mathcal{D}_{g_2^h} \neq \emptyset$. Choose $k \in \mathcal{D}_{g_1^h} \cap \mathcal{D}_{g_2^h}$, so that both $(g_1^h, k) \in \mathcal{D}$ and $(g_2^h, k) \in \mathcal{D}$. But then $(g_1, k^{h^{-1}}) \in \mathcal{D}^G$ and $(g_2, k^{h^{-1}}) \in \mathcal{D}^G$, so $k^{h^{-1}}$ is our required f.

Hence E is $\exists$ definable in the language of groups via the following formula.

$$xEy \leftrightarrow \exists uvwz\ u^v = h \wedge (x, u)^w = (g_1, g_2) \wedge (y, u)^z = (g_1, g_2),$$

where $h \in X_d^G$, $g_1, g_2 \in \mathcal{D}$ are parameters. $\square$

The following theorem follows from the above discussion:

Theorem 2.2.2 *Let $\mathcal{M}$ be an ω-categorical, transitive and homogeneous structure in a relational language which satisfies the hypotheses of Proposition 2.1.9. Then $\mathcal{M}$ has a weak $\forall\exists$ interpretation.*

2.3 Extension lemmas

We produce a range of extension lemmas which will make the Banach-Mazur game described above work for various relational structures. The proofs given here are essentially due to Bernhard Herwig. The motivation in Herwig's work was to obtain a proof of the small index property for the structures treated, by producing an equivalent of Hrushovski's extension lemma for graphs [18] used in [17]. Herwig's proofs cover the extension property for partial isomorphisms without any restriction on the cycle type of the isomorphisms involved. We show how minimal modifications of his proofs yield the extension property for finite partial isomorphisms having a unique fixed point. We shall use Herwig's notation and arguments throughout to prove the fixed point extension property in definition 2.1.7 for a range of different classes of structures.

Herwig's proofs are by induction on the maximal arity k of the relation symbols in the language S concerned, and, later, on the maximal size of certain forbidden configurations. In both cases the induction hypothesis is used by reducing k as follows: for a k-ary relation symbol R we introduce new relation symbols R_a, for all $a \in A$, of arity $k-1$, interpreted as

$$A \models R_a\bar{b} \iff A \models Ra\bar{b}.$$

Once we have done that, we have to change the notion of partial isomorphism. Recall that, if p is a partial isomorphism of A,

$$A \models Ra\bar{b} \iff A \models Ra^p\bar{b}^p$$

for all $a, \bar{b}$ in the domain of p. To express this in the new language we need that

$$A \models R_a\bar{b} \iff A \models R_{a^p}\bar{b}^p$$

so we need to allow for a permutation of the new relation symbols. This is what motivates Herwig's definition of *permorphism*:

Definition 2.3.1 *Let S be a relational language, $\chi \in \mathrm{Sym}(S)$ a permutation mapping every symbol to a symbol of the same arity. Let $\mathcal{A}$ be an S-structure and p a partial injective mapping on $\mathcal{A}$. Then p is a χ-permorphism if for all $r \in \omega$, all r-ary $R \in S$, $a_1, \ldots, a_r \in \mathrm{dom}(p)$:*

$$Ra_1 \cdots a_r \iff R^\chi a_1^p \cdots a_r^p.$$

We shall repeatedly need an easy preliminary combinatorial fact.

Fact 2.3.2 *Let X, Y be finite sets such that $|X| = |Y| \geq 2$. Then there is a fixed-point free bijection $\alpha : X \to Y$.*

Proof Proceed by induction on $|X| = |Y|$. The base case $|X| = 2$ is easy:

- if $X = Y = \{a, b\}$ then take $\alpha = (ab)$;
- if $X = \{a, b\}$ and $Y = \{a, c\}$, take $\alpha = \{\langle a, c\rangle, \langle b, a\rangle\}$;
- if $X = \{a, b\}$ and $Y = \{c, d\}$, take $\alpha = \{\langle a, c\rangle, \langle b, d\rangle\}$.

For the inductive step:
1. If $X \neq Y$, choose $x \in X, y \in Y$ such that $x \neq y$. By the inductive hypothesis, find a fixed-point free bijection $\alpha' : X\backslash\{x\} \to Y\backslash\{y\}$, and define $\alpha := \alpha' \cup \{\langle x, y\rangle\}$.
2. If $X = Y = \{x_1, \ldots, x_n\}$, choose $\alpha := (x_1 x_2 \cdots x_n)$. $\square$

Following [16], we shall produce separate extension lemmas for three different classes of structures:

1. the class of all finite structures in a given finite relational language S;

2. the class of finite K_m-free graphs, for $m \in \omega$;

3. the class of all finite irreflexive structures omitting certain configurations, described in Section 2.3.3.

Case 1. is needed in order to prove the base step in the induction arguments for 2. and 3. The class of K_m-free graphs in 2. will actually turn out to be included in the cases covered by 3. Nevertheless, it is treated separately as a paradigm of the more intricate case 3.

2.3.1 General relational structures

This section will be devoted to proving that the fixed point extension property 2.1.7 holds for a class of structures in a finite relational language, without further restrictions.

Theorem 2.3.3 *Let S be a finite relational language, and let κ be the class of all finite S-structures. Then κ has FEP, the fixed point extension property for partial isomorphisms.*

Theorem 2.3.3 follows from the permorphism version below, which is the base step in the induction argument for graphs (see 2.3.7), and for the more general Lemma 2.3.11. Theorem 2.3.3 is obtained from 2.3.4 by taking all the permorphisms χ_i in the hypothesis of 2.3.4 to be the identity on S.

Lemma 2.3.4 *Let S be a finite relational language, $\chi_1, \dots, \chi_n$ be permutations of S mapping every symbol to a symbol of the same arity, and let $\mathcal{A}$ be a finite S structure. Let $p_1, \dots, p_n$ be partial mappings on $\mathcal{A}$ such that p_i is a χ_i-permorphism and $\mathrm{fix}(p_1) = \mathrm{fix}(p_2) = \cdots = \mathrm{fix}(p_n) = \{d\}$. Then there are a finite S-structure $\mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B}$, and $f_1, \dots f_n \in \mathrm{Sym}(\mathcal{B})$ such that f_i is a χ_i-permorphism, $p_i \subseteq f_i$, and $\mathrm{fix}(f_i) = \mathrm{fix}(p_i)$ for $i = 1, \dots, n$. Moreover, $\mathcal{B}$ and $f_1, \dots, f_n$ can be chosen so that:*

1. *$\forall b \in \mathcal{B}\ \exists f \in \langle f_1, \ldots, f_n \rangle$ s.t. $b^f \in \mathcal{A}$;*

2. *for every r-ary relation symbol R and $b_1, \ldots, b_r \in \mathcal{B}$, if $\mathcal{B} \models Rb_1 \cdots b_r$, then there is $f \in \langle f_1, \ldots, f_n \rangle$ such that $b_i^f \in \mathcal{A}$, $i = 1 \ldots r$;*

3. *if S contains relations of arity greater than 1, then: for all $f \in \langle f_1, \ldots, f_n \rangle$ and $a, b \in \mathcal{A}$ with $a^f = b$, there are $t \in \omega$, $i_1, \ldots i_t \in \{1, \ldots, n\}$ and $\epsilon_1, \ldots, \epsilon_t \in \{-1, 1\}$ such that $a^{p_{i_1}^{\epsilon_1} \cdots p_{i_t}^{\epsilon_t}} = b$ and $f_{i_1}^{\epsilon_1} \cdots f_{i_t}^{\epsilon_t} = f$.*

Proof We proceed by induction on the maximal arity of the symbols in S. We need to set up the notation first. For each $i = 1, \ldots, n$, let $D_i = \mathrm{dom}(p_i), D'_i := \mathrm{ran}(p_i)$. Moreover:

- by Δ-tp$(c/\mathcal{A})$ we shall denote the positive atomic type without equality of c over $\mathcal{A}$, for $c \in \mathcal{C} \supseteq \mathcal{A}$. By a Δ-type over $\mathcal{A}$ we shall mean a positive atomic type without equality over $\mathcal{A}$;
- if p is a Δ-type over $\mathcal{A}$ and $\mathcal{A}_0 \subseteq \mathcal{A}$, then $p \restriction_{A_0} := \{Rx\bar{a} \in p \mid \bar{a} \in \mathcal{A}_0\}$;
- if $\psi = Rx\bar{a}$, with $\bar{a} \in D_i$, then $\psi^{p_i} := R^{\chi_i} x \bar{a}^{p_i}$;
- if p is a Δ-type over D_i, then $p^{p_i} := \{\psi^{p_i} : \psi \in p\}$ is a Δ-type over D'_i.

We keep Herwig's notation throughout in order to make it easy to refer back to the original proof. Note, however, the following notational clashes:
- p indicates a Δ-type and p_i a mapping on $\mathcal{A}$;
- c_0 is a constant in $\mathbb{N}$, and c sometimes is used to denote a general element of $\mathcal{A}$;
- χ_i is a permutation of the symbols in the language S, ϕ_i will be a permutation of a structure $\mathcal{F}$ (see below).

Base case Suppose that the maximal arity of the symbols in S is 1, so that S only contains unary predicates, and let $\langle \mathcal{A}, p_1, \ldots, p_n \rangle$ be as in the hypothesis of the Lemma. Then a Δ-type will be a set of formulae of the form Rx for $R \in S$.

For a Δ-type t, let c_t be the number of realisations of t in $\mathcal{A}$, and let

$$k = \max\{\max\{c_t : t \text{ is a } \Delta\text{-type}\}, 3\}.$$

Let $\mathcal{D}$ be the set of all Δ-types. We can impose an S-structure on $\mathcal{D}$ as follows: for $R \in S$ and $t \in \mathcal{D}$ define

$$\mathcal{D} \models Rt \iff Rx \in t.$$

Then there is a homomorphism $\mathcal{A} \to \mathcal{D}$ defined by $a \in \mathcal{A} \to \Delta\text{-tp}(a)$: for any $a \in \mathcal{A}$, $\mathcal{A} \models Ra \quad \Rightarrow \quad Rx \in \Delta\text{-tp}(a) \quad \Rightarrow \quad \mathcal{D} \models R(\Delta\text{-tp}(a))$.

Take $\mathcal{B}$ to be the disjoint union of k copies of $\mathcal{D}$. Then $\mathcal{A}$ embeds in $\mathcal{B}$ in the obvious way: if $a_1, \ldots, a_m$ are such that $\Delta\text{-tp}(a_1) = \Delta\text{-tp}(a_2) = \ldots = \Delta\text{-tp}(a_m)$, for $m \leq k$, and $t^1, \ldots, t^k$ are the k copies of t in $\mathcal{B}$, then $a_j \to t^j$.

We can now extend p_i to a permorphism f_i of $\mathcal{B}$ by mapping t^j, $j = 1, \ldots, k$ to any of the k copies of t^{p_i}, say

$$(t^j)^{f_i} := (t^{p_i})^j.$$

Since $k \geq 3$ by hypothesis, we can arrange for f_i to have a single fixed point: when $t = t^{p_i}$, renaming the indexes if necessary, we can put $(t^1)^{p_i} := (t^1)$, and use Lemma 2.3.2 to get $(t^j)^{f_i} \neq t^j$ for all $j \neq 1$.

Conditions 1. and 2. are achieved as follows: if necessary, replace $\mathcal{B}$ by $\{a^f : a \in \mathcal{A}, f \in \langle f_1, \ldots, f_n \rangle\}$, and restrict the interpretation of a predicate R to all elements of the form a^f, where $a \in \mathcal{A}$, $f \in \langle f_1, \ldots, f_n \rangle$.

Inductive step We start by building a finite S-structure $\mathcal{C}$ extending $\mathcal{A}$ such that:

1. there is $c_0 \in \omega$ such that for every Δ-type p over $\mathcal{A}$

$$|\{c \in \mathcal{C} \mid \Delta\text{-tp}(c/\mathcal{A}) = p\}| = c_0;$$

2. there are $h_1, \ldots, h_n \in \mathrm{Sym}(\mathcal{C})$ such that $p_i \subseteq h_i$, $\mathrm{fix}(p_i) = \mathrm{fix}(h_i)$, and for all $\bar{a} \in D_i$, $b \in \mathcal{C}, R \in S$: $Rb\bar{a} \iff R^{\chi_i} b^{h_i} \bar{a}^{p_i}$.

The only difference with Herwig's proof here is the requirement that $\mathrm{fix}(p_i) = \mathrm{fix}(h_i)$ in 2.

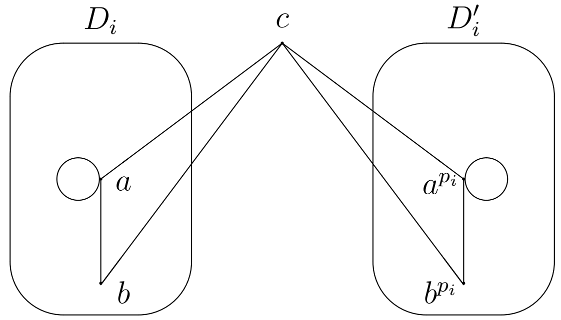
Let p be a Δ-type over $\mathcal{A}$, then $c_p := \{c \in \mathcal{A} \mid \Delta\text{-tp}(c/\mathcal{A}) = p\}|$. Let

$$c_0 := \max\{c_p \mid p \text{ is a } \Delta\text{-type over } \mathcal{A}\} + 2.$$

The constant 2 is added in order to guarantee the extra condition $\mathrm{fix}(p_i) = \mathrm{fix}(h_i)$, as will be clear later. Now we proceed exactly as in Herwig's proof: for each Δ-type p over $\mathcal{A}$ we add $(c_0 - c_p)$ new points c with $\Delta\text{-tp}(c/\mathcal{A}) = p$. Let $\mathcal{C}$ be $\mathcal{A}$ together with the new points. There will be no structure among the added points: every instance of a relation in $\mathcal{C}$ will involve at most one point of $\mathcal{C} \setminus \mathcal{A}$.

Pick p_i, and any Δ-type p over D_i. Let $p' := p^{p_i}$, and $C_i^p := \{c \in \mathcal{C} \mid \Delta\text{-tp}(c/D_i) = p\}$ (and likewise $C_i^{p'} = \{c \in \mathcal{C} \mid \Delta\text{-tp}(c/D_i') = p'\}$). We have $|C_i^p| = |C_i^{p'}|$ via an inclusion-exclusion argument (see [16] for the proof). Now we can choose h_i to map C_i^p bijectively to $C_i^{p'}$. Note that if we do this, the condition that $Rb\bar{a} \iff R^{\chi_i} b^{h_i} \bar{a}^{p_i}$ is achieved for all $\bar{a} \in D_i$, $b \in \mathcal{C}, R \in S$. In Herwig's version of the lemma, we have complete freedom on how we choose our bijection h_i, modulo the constraint that it should extend p_i. In our case, we have to ensure that h_i has no new fixed points. By our choice of c_0, we have that $|C_i^p \setminus \mathcal{A}| \geq 2$, and so $|C_i^p \setminus D_i| \geq 2$, and likewise for $C_i^{p'} \setminus D_i'$. We can then choose $h_i := p_i \cup \alpha_i$, where $\alpha_i : C_i^p \setminus D_i \to C_i' \setminus D_i'$ is a bijection. By 2.3.2, α_i can be chosen to be fixed-point free, so that $\mathrm{fix}(h_i) = \mathrm{fix}(p_i)$, as required.

Adding the constant 2 to $\max\{c_p\}$ ensures that there is always room to extend p_i to a permutation h_i with no further fixed points: each set on which h_i is to be defined contains at least 2 new points, hence 2.3.2 can be applied. In particular, situations of the following kind are avoided: suppose $\mathcal{A}$ is given, where the language contains a single binary symmetric relation R, and suppose $\max\{c_p\} = 2$, and for some partial permorphism p_i and $a, b, c \in \mathcal{A}$ the following configuration is given, where edges among points represent instances of R:



Here $p = \Delta\text{-tp}(c/D_i) = \{Rxa, Rxb\} = \Delta\text{-tp}(a/D_i)$, therefore

$$|\{d \in \mathcal{A} : \Delta\text{-tp}(d/D_i) = p\}| = 2.$$

If we do not require that new realisations of $\Delta\text{-tp}(c/D_i)$ should be added to the structure, this configuration would force the extension h_i of p_i to fix c.

We now produce an extension of $\mathcal{C}$ in a new language S'. Let s be the maximal arity of the symbols in S, S_s be the set of symbols of S of arity s, and define

$$S' := S \setminus S_s \cup \{R_c : R \in S_s, c \in \mathcal{C}\}.$$

The maximal arity in S' is $s-1$. We can regard $\mathcal{A}$ as a S' structure, say $\mathcal{A}'$, with the R_c interpreted in the obvious way:

$$\mathcal{A}' \models R_c\bar{a} \iff \mathcal{C} \models Rc\bar{a}$$

for all $c \in \mathcal{C}, \bar{a} \in \mathcal{A}$. We then define χ'_i on $\{R_c \mid c \in \mathcal{C}, R \in S_s\}$ by $R_c^{\chi'_i} := R_{c^{h_i}}^{\chi_i}$, and $\chi'_i = \chi_i$ on $S \setminus S_s$. In this way, p_i is a χ'_i-permorphism on $\mathcal{A}'$. By induction, we get a finite S'-structure $\mathcal{F}$, $\mathcal{A}' \subseteq \mathcal{F}$, and $\phi_1, \dots, \phi_n \in \mathrm{Sym}(\mathcal{F})$ such that $p_i \subseteq \phi_i$, ϕ_i is a χ'_i-permorphism and $\mathrm{fix}(\phi_i) = \mathrm{fix}(p_i)$. We also get the additional properties 1. and 2. in the statement of the lemma.

Now let $\gamma_i = (\chi_i, \phi_i, h_i) \in \mathrm{Sym}(S) \times \mathrm{Sym}(\mathcal{F}) \times \mathrm{Sym}(\mathcal{C})$, and define $\Gamma := \langle \gamma_i,\ i = 1, \dots, n\rangle$, the subgroup generated by the γ_i. The components of an element $\gamma \in \Gamma$ will be denoted by (χ, ϕ, h) throughout. Note that Γ acts on S (by $R^\gamma := R^\chi$), on $\mathcal{F}$ (by $c^\gamma := c^\phi$) and on $\mathcal{C}$ (by $c^\gamma := c^h$).

Consider now the set $\mathcal{A} \times \Gamma$, and define $E \subseteq (\mathcal{A} \times \Gamma) \times (\mathcal{A} \times \Gamma)$ by

$$(a^{p_i}, \gamma)E(a, \gamma_i \gamma)$$

and let $\equiv$ be the reflexive, symmetric and transitive closure of E, so that $\equiv$ is an equivalence relation on $\mathcal{A} \times \Gamma$. The idea is that we are extending $\mathcal{A}$ by adding all the images under Γ of elements of $\mathcal{A}$. Quotienting by $\equiv$ enables us to collapse different ways of writing the same image into the same equivalence class.

The required extension of $(\mathcal{A}, p_1, \ldots, p_n)$ will then be $(\mathcal{A} \times \Gamma / \equiv, f_1, \ldots, f_n)$, where the f_i act like the γ_i.
The following hold and are proved in [16]:

1. if $(a, (\chi, \phi, h)) \equiv (a', (\chi', \phi', h'))$ then $a^\phi = (a')^{\phi'}$ and $a^h = (a')^{h'}$,
2. if $R \in S \cup S'$ is r-ary and $(a_1, \gamma) \equiv (a_1', \gamma'), \ldots, (a_r, \gamma) \equiv (a_r', \gamma')$ then $R^{\gamma^{-1}} a_1 \cdots a_r \iff R^{(\gamma')^{-1}} a_1' \cdots a_r'$;
3. $R^\gamma (a_1, \gamma)/\equiv \cdots (a_r)/\equiv \iff Ra_1 \cdots a_r$;
4. if $(a, \gamma_1) \equiv (b, \gamma_2)$ then there are $t \in \omega$, $p_{i_1}, \ldots, p_{i_t}$ and $\epsilon_1, \ldots, \epsilon_t \in \{1, -1\}$ such that $b^{p_{i_1} \cdots p_{i_t}} = a$ and $\gamma_2 = \gamma_{i_1}^{\epsilon_1} \cdots \gamma_{i_t}^{\epsilon_t} \gamma_1$.

The map $i : \mathcal{A} \hookrightarrow \mathcal{A} \times \Gamma / \equiv$ defined by $i(a) = (a, 1)/\equiv$ is an embedding of S-structures. We then define the χ_i-permorphism f_i by

$$((a, \gamma)/\equiv)^{f_i}) := (a, \gamma\gamma_i)/\equiv .$$

In fact f_i satisfies all the properties required in the lemma. The proof is contained in [16]. We are only left to check that $\mathrm{fix}(f_i) = \mathrm{fix}(p_i)$. So suppose $(a, \gamma)/\equiv \in \mathcal{A} \times \Gamma/\equiv$ is such that $((a, \gamma)/\equiv)^{f_i} = (a, \gamma)/\equiv$. Then $(a, \gamma\gamma_i) \equiv (a, \gamma)$, i.e. $(a, (\chi\chi_i, \phi\phi_i, hh_i)) \equiv (a, (\chi_i, \phi_i, h_i))$. By property 1. above, $a^{\phi\phi_i} = a^{\phi_i}$ and $a^{hh_i} = a^{h_i}$. We know that ϕ_i, h_i have a unique fixed point. It follows that $a^\phi = a^h = \mathrm{fix}(p_i)$, so that $a^\phi = a^h \in \mathcal{A}$. We need to show that $i(a^\phi) = (a, \gamma)/_\equiv$. But $\gamma \in \langle \gamma_i,\ i = 1, \ldots, n \rangle$, so there are $i_1, \ldots, i_t$ and $\epsilon_1, \ldots, \epsilon_t \in \{1, -1\}$ such that $\gamma = \gamma_{i_1}^{\epsilon_1} \cdots \gamma_{i_t}^{\epsilon_t} = (\chi_{i_1}, \phi_{i_1}, h_{i_1}) \cdots (\chi_{i_t}, \phi_{i_t}, h_{i_t})$. Then $(a^h, 1) = (a^\phi, 1) = (a^{p_{i_1} \cdots p_{i_t}}, 1)$, and $(a^{p_{i_1} \cdots p_{i_t}}, 1) \equiv (a, \gamma_{i_1} \cdots \gamma_{i_t})$, so $(a^\phi, 1) \equiv (a, \gamma)$ (by 4. above), as required. $\square$

2.3.2 K_m-free graphs

We sketch Herwig's argument for the extension lemma for K_m-free graphs, that is, graphs which do not embed the complete graph on m vertices. Our aim is to prove

Theorem 2.3.5 *Let $m \in \mathbb{N}$, and let κ be the class of finite K_m-free graphs. Then κ has FEP, the fixed point extension property for finite partial isomorphisms.*

As in the previous section, the result is proved from a permorphism version, Lemma 2.3.7 below. The proof of the permorphism lemma is again by induction on m, and once again the idea is to reduce a K_{m+1}-freeness condition to a K_m-freeness condition. Given a graph $\mathcal{A}$, this is achieved by introducing a new unary predicate - a colour - U_a for all $a \in \mathcal{A}$, to be interpreted as

$$\mathcal{A} \models U_a(b) \iff \mathcal{A} \models aRb,$$

where R is the graph relation. This will enable us to express the K_{m+1}-freeness condition as follows: a graph is K_{m+1}-free if and only if it does not contain a K_m graph whose vertices $a_1, \ldots a_m$ have all the same colour U_a for some $a \in \mathcal{A}$.

Suppose we are given a K_{m+1}-free graph $\mathcal{A}$, and partial isomorphisms $p_1, \ldots, p_n$ of $\mathcal{A}$ such that $\text{fix}(p_1) = \ldots = \text{fix}(p_n) = \{d\}$. The aim is to find extensions $\mathcal{B}$ of $\mathcal{A}$ and f_i of p_i such that $f_i \in \text{Aut}(\mathcal{B})$ and $\text{fix}(f_i) = \text{fix}(p_i)$. We treat $\mathcal{A}$ as a coloured graph, but in the new language $L = \{R\} \cup \{U_a : a \in \mathcal{A}\}$ the p_i are partial permorphisms, rather than partial isomorphisms:

$$\mathcal{A} \models U_a(b) \iff \mathcal{A} \models aRb \iff \mathcal{A} \models a^{p_i} R b^{p_i} \iff \mathcal{A} \models U_{a^{p_i}}(b^{p_i}),$$

so each p_i is in fact a permorphism with respect to the permutation χ_i of L defined by $\chi_i(U_a) := U_{a_i^p}$ for all $a \in \mathcal{A}$, and $\chi_i(R) := R$.

As in the previous proof, we build a "type realising" extension $\mathcal{C}$ of $\mathcal{A}$. We then expand L to $L' := L \cup \{U_c : c \in \mathcal{C}\}$, and consider $\mathcal{A}$ as an L' structure. In order to make the inductive hypothesis apply, we need $\mathcal{A}$ to satisfy (unicoloured-K_m-freeness) with respect to the new colours. Note that the (unicoloured-K_{m+1})-freeness of $\mathcal{A}$ with respect to the colours in L is not equivalent to its (unicoloured-K_m)-freeness with respect to the colours in L'. Herwig finds the following equivalence instead:

$\mathcal{A}$ is (unicoloured-K_{m+1})-free with respect to colours in $L \iff$ there are no colour $V \in L$, $a \in \mathcal{A}$ of colour V, a colour $V_a \in L'$ and a copy of K_m which is coloured with both V and V_a.

To see why the equivalence holds, suppose there are $V \in L$, $V_a \in L'$, $a \in \mathcal{A}$ such that Va holds, and $a_1, \ldots, a_m \in \mathcal{A}$ such that $a_i R a_j$, Va_i and $V_a a_j$ for all $i = 1, \ldots, m, i \neq j$. Then clearly $a, a_1, \ldots, a_m$ is a copy of K_{m+1} of colour V, so $\mathcal{A}$ is not K_{m+1} free with respect to the colours in L. This proves the $\Rightarrow$ direction. For $\Leftarrow$, a copy of K_{m+1}, coloured with a colour $V \in L$, gives the required $V, V_a, a, a_1, \ldots a_m$ (take a to be any of the vertices in the given unicoloured K_{m+1} graph).

So the original graph $\mathcal{A}$ satisfies the (unicoloured-K_m)-freeness condition in L' which is equivalent to (unicoloured-K_{m+1})-freeness in L. The inductive hypothesis yields an extension $\mathcal{B}$ of $\mathcal{A}$ which also satisfies this condition, which will turn out to be enough to guarantee the (unicoloured-K_{m+1})-freeness of $\mathcal{B}$ with respect to L.

These considerations justify Herwig's definition of a *critical colouring*:

Definition 2.3.6 *Let $\mathfrak{U}^1, \ldots, \mathfrak{U}^r$ be disjoint sets of unary predicates, or **colours**, and let $\mathfrak{U} := \bigcup_{1 \leq j \leq r} \mathfrak{U}^j$. Let $\mathcal{A}$ be an $\mathfrak{U} \cup \{R\}$ structure, $V \in \mathfrak{U}$ and $a \in \mathcal{A}$. Then we say that a has colour V whenever $\mathcal{A} \models Va$.*

*A $\mathfrak{U}$-**graph** is a $\mathfrak{U} \cup \{R\}$ structure $\mathcal{A}$ such that the reduct of $\mathcal{A}$ to $\{R\}$ is a graph.*

*Let $\mathfrak{U}_c \subset \mathfrak{U}^1 \times \cdots \times \mathfrak{U}^r$. We shall call $\mathfrak{U}_c$ the set of **critical colourings**. Then $\mathcal{A}$*

is said to be $\mathfrak{U}_c$*-*K_m***-free*** *if there are no colouring* $(V_1, \dots, V_r) \in \mathfrak{U}_c$ *and vertices* $a_1, \dots, a_m \in \mathcal{A}$ *such that* $a_i R a_j$ *and* $a_i \in V_k$ *for all* $i, j \in \{1, \dots, m\}, i \neq j$ *and* $k \in \{1, \dots, r\}$.

The $\mathfrak{U}_i$ are intended to represent sets of old and new colours.

We give a sketch of Herwig's argument, with an indication of the basic modification needed to obtain our required cycle type for the isomorphisms.

Lemma 2.3.7 *Let* $m \geq 1$*, and let* $\mathfrak{U}^1, \dots, \mathfrak{U}^r$ *be disjoint sets of colours,* $r \geq 0$*. Let* $\mathfrak{U} := \bigcup_{1 \leq j \leq r} \mathfrak{U}^j$*. Let* $\chi_i^j \in \mathrm{Sym}(\mathfrak{U}^j)$ *for* $i \in \{1, \dots, n\}$*,* $j \in \{1, \dots, r\}$*, and* $S := \mathfrak{U} \cup \{R\}$*, where* R *is the graph relation. Define* $\chi_i := \bigcup_{0 \leq j \leq r} \chi_i^j \in \mathrm{Sym}(S)$*, where* $\chi_i^0(R) = R$*, and let* $\mathfrak{U}_c \subset \mathfrak{U}^1 \times \cdots \times \mathfrak{U}^r$ *be the set of critical colourings. Suppose further that* $\mathfrak{U}_c$ *is invariant under each* χ_i*.*

Let $\mathcal{A}$ *be a finite* $\mathfrak{U}_c$*-*K_m*-free* $\mathfrak{U}$*-graph, and suppose* $p_1, \dots, p_n$ *are partial permorphisms of* $\mathcal{A}$ *such that* $\mathrm{fix}(p_1) = \cdots = \mathrm{fix}(p_n) = \{c\}$*. Then there are a finite* $\mathfrak{U}$*-graph* $\mathcal{B}$ *such that* $\mathcal{A} \subset \mathcal{B}$*,* $\mathcal{B}$ *is* $\mathfrak{U}_c$*-*K_m*-free, and* $f_1, \dots, f_n \in \mathrm{Sym}(\mathcal{B})$ *such that:*

1. $p_i \subseteq f_i$*,* $i = 1, \dots, n$*;*
2. f_i *is a* χ_i*-permorphism,* $i = 1, \dots, n$*;*
3. $\mathrm{fix}(f_i) = \mathrm{fix}(p_i)$*,* $i = 1, \dots, n$*;*
4. $\forall b \in \mathcal{B}\ \exists f \in \langle f_1, \dots, f_n \rangle$ *s.t.* $b^f \in \mathcal{A}$*;*
5. *for all* $a, b \in \mathcal{B}$*, if* $\mathcal{B} \models aRb$*, then there is* $f \in \langle f_1, \dots, f_n \rangle$ *such that* $a^f, b^f \in \mathcal{A}$*;*
6. *if* S *contains relations of arity greater than 1, then: for all* $f \in \langle f_1, \dots, f_n \rangle$ *and* $a, b \in \mathcal{A}$ *with* $a^f = b$*, there are* $t \in \omega$*,* $p_{i_1}, \dots p_{i_t} \in \{p_1, \dots, p_n\}$ *and* $\epsilon_1, \dots, \epsilon_t \in \{-1, 1\}$ *such that* $a^{p_{i_1}^{\epsilon_1} \cdots p_{i_t}^{\epsilon_t}} = b$ *and* $f_{i_1}^{\epsilon_1} \cdots f_{i_t}^{\epsilon_t} = f$*.*

Proof We proceed by induction on m. The base case rests on 2.3.4, where the maximal arity in the language S is 1, and, with our version of the lemma, it is exactly as in [16]: suppose $m = 1$, i.e. $\mathcal{A}$ does not contain a point a such that, for some $(V_1, \ldots, V_r) \in \mathfrak{U}_c$, $\mathcal{A} \models V_j a$ for all $j = 1, \ldots, r$. By 2.3.4, there is $\mathcal{B}$ extending $\mathcal{A}$ and $f_1, \ldots, f_n$ such that $f_i \supseteq p_i$ and $\mathrm{fix}(f_i) = \mathrm{fix}(p_i)$ for all $i = 1, \ldots, n$. We claim that $\mathcal{B}$ is $\mathfrak{U}_c$-K_1-free. Suppose for a contradiction that there are $b \in \mathcal{B}$ and $(V_1, \ldots, V_r) \in \mathfrak{U}_c$ such that $\mathcal{B} \models V_j b$ for all $j = 1, \ldots, r$. By property 1. of the f_i, pick $f \in \langle f_1, \ldots, f_n \rangle$ with $b^f \in \mathcal{A}$ and suppose f is a χ-permorphism for some $\chi \in \langle \chi_1, \ldots, \chi_n \rangle$. Then $(V_1^\chi, \ldots, V_r^\chi) \in \mathfrak{U}_c$ by χ_i-invariance of $\mathfrak{U}_c$, hence $\mathcal{A} \models V_j^\chi b^f$ for all $j = 1, \ldots, r$, which contradicts the $\mathfrak{U}_c$-K_1-freeness of $\mathcal{A}$.

For the inductive step we suppose the result holds for $\mathfrak{U}_c$-K_m-free graphs, and we consider a $\mathfrak{U}_c$-K_{m+1}-free $\mathcal{A}$ and partial permorphisms $p_1, \ldots, p_n$ as in the hypothesis. A Δ-type p over $\mathcal{A}$ will again be a positive atomic type over $\mathcal{A}$ without equality, i.e. a set of formulae of the form Vx, $V \in \mathfrak{U}$ and xRa, $a \in \mathcal{A}$. Then

$$\Delta\text{-tp}(c/\mathcal{A}) := \{xRa \ : \ \mathcal{A} \models cRa\} \cup \{Vx \ : \ \mathcal{A} \models Vc\}.$$

Let $\mathrm{Par}(p) \subseteq \mathcal{A}$ be the set of parameters appearing in p, and $\mathfrak{U}(p) := \{V \in \mathfrak{U} \ : \ Vx \in p\}$. Then p is said to be **realisable** if it does not realise a $\mathfrak{U}_c$-K_{m+1} graph, i.e. if there are no $a_1, \ldots, a_m \in \mathrm{Par}(p)$ and $V_1, \ldots, V_r \in (\mathfrak{U}(p))^r \cap \mathfrak{U}_c$ such that $a_i R a_j$ for all $i, j \in \{1, \ldots, m\}, i \neq j$ and $V_k a_i$ for all $k \in \{1, \ldots, r\}, i \in \{1, \ldots m\}$.

We are going to build an extension $\mathcal{C}$ of $\mathcal{A}$ where all realisable types over $\mathcal{A}$ have a fixed numbers of realisations, which depends on the number of parameters appearing in the type. The claim is that

1. there is a $\mathfrak{U}_c$-K_{m+1}-free graph $\mathcal{C} \supseteq A$ and for every $t \in \{1, \ldots |\mathcal{A}|\}$ a constant $c_t \in \omega$ such that for every Δ-type p over $\mathcal{A}$
$$|\{c \in \mathcal{C} \ : \ \Delta\text{-tp}(c/\mathcal{A}) \supseteq p, \mathfrak{U}(c) = \mathfrak{U}(p)\}| = \begin{cases} c_t, \ t = |\mathrm{Par(p)}|, & \text{if } p \text{ is realisable} \\ \quad 0 & \text{otherwise} \end{cases}$$
2. there are bijections $h_1, \ldots, h_n \in \mathrm{Sym}(\mathcal{C}), h_i \supseteq p_i$, $\mathrm{fix}(h_i) = \mathrm{fix}(p_i) = \{d\}$, and such that for every $V \in \mathfrak{U}, b \in \mathcal{C}, a \in D_i$ and $i \in \{1, \ldots, n\}$:
$$Vb \iff V^{\chi_i} b^{h_i}, \text{ and}$$

$$aRb \iff a^{p_i} R b^{h_i}.$$

Let $T = |\mathcal{A}|$. We shall produce a chain $\mathcal{A} = \mathcal{C}_T \subseteq \mathcal{C}_{T-1} \subseteq \cdots \subseteq \mathcal{C}_0 = \mathcal{C}$ and constants $c_T, \ldots, c_0$ such that for all Δ-types p over $\mathcal{A}$ with $|\mathrm{Par}(p)| \geq t$:

$$|\{d \in \mathcal{C}_t : d \models p, \mathfrak{U}(d) = \mathfrak{U}(p)\}| = c_{|Par(p)|},$$

so that at stage t all types over $\mathcal{A}$ with a big enough set of parameters satisfy the requirement. We let $\mathcal{C}_T = \mathcal{A}$ and $c_T = 0$ (no types over $\mathcal{A}$ with exactly $|\mathcal{A}|$ parameters are realised in $\mathcal{A}$, because R is irreflexive).

We suppose inductively that $\mathcal{C}_T, c_T, \mathcal{C}_{T-1}, c_{T-1}, \ldots, \mathcal{C}_t, c_t$ have been constructed. We build $\mathcal{C}_{t-1}$ by adding a suitable number of realisations of each type p over $\mathcal{A}$ with $|\mathrm{Par}(p)| = t-1$. For such a type let

$$c_p := |\{c \in \mathcal{C}_t \ : \ \Delta\text{-tp}(c/\mathcal{A}) \supseteq p, \ \mathfrak{U}(c) = \mathfrak{U}(p)\}|, \text{ and}$$

$$c_{t-1} = \max\{c_p : p \text{ is a } \Delta\text{-type over } \mathcal{A}, \ |\mathrm{Par}(p)| = t-1\}.$$

Now for every realisable p with $|\mathrm{Par}(p)| = t-1$ we add to $\mathcal{C}_t$ exactly $c_{t-1} - c_p$ new realisations of p. No new instances of R appear among the added points. Via an inclusion/excusion argument, we get that $\mathcal{C}$ is such that for all Δ-types p over $\mathcal{A}$:

$$|\{c \in \mathcal{C} \ : \ \Delta\text{-tp}(c/D_i) = p\}| = |\{c \in \mathcal{C} \ : \ \Delta\text{-tp}(c/D_i') = p^{p_i}\}|,$$

and, as in 2.3.4, we choose h_i to be any bijection between the first and the second set extending p_i.

The rest of Herwig's argument goes through unchanged: we introduce new colours $\mathfrak{U}^{r+1} = \{U_d^{r+1} : d \in \mathcal{C}\}$, where U_d^{r+1} is a new unary relation symbol for each $d \in \mathcal{C}$. We define $L' := L \cup \mathfrak{U}^{r+1}$, $\mathfrak{U}' := \mathfrak{U} \cup \mathfrak{U}^{r+1}$ and we extend the χ_i to bijections χ_i' of L' as follows:

1. let $\chi_i^{r+1} \in \mathrm{Sym}(\mathfrak{U}^{r+1})$ be defined by $(U_d^{r+1})^{\chi_i^{r+1}} := U_{d^{h_i}}^{r+1}$;
2. let $\chi' := \chi \cup \chi_i^{r+1}$.

The colours in $\mathfrak{U}^{r+1}$ are interpreted in $\mathcal{A}$ in the obvious way:

$$\mathcal{A} \models U_d^{r+1}(a) \iff \mathcal{C} \models dRa,$$

so that $\mathcal{A}$ is a $\mathfrak{U}'$-graph. We have a new set $\mathfrak{U}'_c \subseteq \mathfrak{U}^1 \times \cdots \times \mathfrak{U}^{r+1}$ defined by

$$(V_1, \ldots, V_r, U_d^{r+1}) \in \mathfrak{U}'_c \iff (V_1, \ldots, V_r) \in \mathfrak{U}_c \text{ and } \mathcal{C} \models V_j(d), 1 \leq j \leq r.$$

See Herwig for proofs of the following claims:

- p_i is a χ'_i permorphism;
- $\mathcal{A}$ is $\mathfrak{U}'_c$-K_m-free.

Then the inductive hypothesis applies to $\langle \mathcal{A}, p_1, \ldots, p_n \rangle$, with $\mathcal{A}$ a $\mathfrak{U}'_c$-K_m-free graph, so there is a finite $\mathfrak{U}'_c$-K_m-free graph $\mathcal{B}$ extending $\mathcal{A}$ and permorphisms $f_1, \ldots, f_n \in \mathrm{Sym}(\mathcal{B})$ extending $p_1, \ldots, p_n$ with the properties listed in the hypothesis.

If we consider $\mathcal{B}$ as a $\mathfrak{U}$-graph, then $\mathcal{B}$ is $\mathfrak{U}_c$-K_{m+1}-free, and the f_i have the required properties. The proof is exactly as in [16]. □

2.3.3 A more general case

Herwig's most general lemma concerns a class of structures which includes hypergraphs and K_n-free graphs, treated in the previous sections, as well as Henson digraphs, which can be seen as directed graphs omitting certain sets of tournaments.

We give the definition and a description of the class treated by Herwig, and state the extension lemma for this class, modified to our requirements. We shall only comment briefly on the changes needed for Herwig's proof to work for our version.

Let S be a relational language, and S_k the set of k-ary relation symbols in S. Let us recall Herwig's definitions and notation:

Definition 2.3.8 *1. An S-structure $\mathcal{L}$ is a* ***link structure*** *if $|\mathcal{L}| = 1$ or $\mathcal{L} = \{a_1, \ldots, a_k\}$ and there is $R \in S_k$ such that $\mathcal{L} \models Ra_1 \ldots a_k$.*
2. $\mathcal{L}$ is ***irreflexive*** *if for all $k, R \in S_k, a_1, \ldots, a_k$, $\mathcal{L} \models Ra_1 \ldots a_k$ implies $a_i \neq a_j$*

for all $i \neq j$.

3. Let $\mathfrak{L}$ be a set of link structures. An S-structure $\mathcal{L}$ has ***link type*** *$\mathfrak{L}$ if $\mathfrak{L}$ contains an isomorphic copy of any substructure of $\mathcal{L}$ which is a link structure.*

4. A map $\rho : \mathcal{T} \to \mathcal{A}$, where $\mathcal{T}$ and $\mathcal{A}$ are S-structures, is a ***weak homomorphism*** *if for all $k, R \in S_k, a_1, \ldots, a_k$, $\mathcal{T} \models Ra_1 \ldots a_k \Rightarrow \mathcal{A} \models Ra_1^\rho \ldots a_k^\rho$. If ρ is a weak homomorphism, we shall write $\mathcal{T} \to_w \mathcal{A}$.*

5. Let $\mathfrak{F}$ be a set of finite S-structures. Then an S-structure $\mathcal{A}$ is ***$\mathfrak{F}$-free*** *if there are no $\mathcal{T} \in \mathfrak{F}$ and $\rho : \mathcal{T} \to_w \mathcal{A}$.*

6. An S-structure $\mathcal{A}$ is a ***packed structure*** *if for any $a_1, a_2 \in \mathcal{A}$ there is a link structure $\mathcal{L}$ with $a_1, a_2 \in \mathcal{L}$.*

Examples of a packed structure are tournaments and complete graphs. We are interested in classes of structures which omit certain packed structures:

Notation 2.3.9 *Let $\mathfrak{L}$ be a set of link structures and $\mathfrak{F}$ a set of finite S-structures, then $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ will denote the class of all finite S-structures which are $\mathfrak{F}$-free and of link type $\mathfrak{L}$, and $\mathfrak{K}_{\mathfrak{F}}$ will denote all $\mathfrak{F}$-free irreflexive S-structures.*

Among the examples that can be expressed as structures in classes of the form $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ there are:

- k-hypergraphs, with take $\mathfrak{F} = \emptyset$;
- K_m-free graphs, with $\mathfrak{F}$ containing the complete graph on m vertices;
- Henson digraphs (see Evans [12] for a description of how to view these as a class in this form);
- the arity k analogues of triangle free graphs, namely, the k-hypergraphs not admitting a $k+1$ set all of whose k-tuples are hyperedges.

Henson digraphs and K_m-free graphs are also handled by Rubin, as they are in fact simple on Rubin's definition of simple ([25], §3).

A pivotal observation of Herwig's is that $\mathfrak{K}_{\mathfrak{F}}$ and $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ have the free amalgamation property.

Theorem 2.3.10 *Let S be a finite relational language, $\mathfrak{F}$ a set of finite S structures which are irreflexive and packed, $\mathfrak{L}$ a set of irreflexive link structures. Then $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ has FEP, the fixed point extension property for finite partial isomorphisms.*

The theorem is proved from the following permorphism version:

Lemma 2.3.11 *Let S be a finite relational language, let $\chi_1, \ldots, \chi_n \in \mathrm{Sym}(S)$ be arity preserving permutations. Let $\mathfrak{F}$ be a family of finite irreflexive packed S-structures invariant under χ_i. Let $\mathcal{A} \in \mathfrak{K}_{\mathfrak{F}}$ be finite, and $p_1, \ldots, p_n$ be partial injective maps on $\mathcal{A}$ such that p_i is a χ_i-permorphism, and $\mathrm{fix}(p_1) = \mathrm{fix}(p_2) = \ldots = \mathrm{fix}(p_n) = \{d\}$. Then there exists a finite $\mathcal{B} \in \mathfrak{K}_{\mathfrak{F}}$ and $f_1, \ldots, f_n \in \mathrm{Sym}(\mathcal{B})$ such that:*

1. $\mathcal{A} \subseteq \mathcal{B}$;
2. $p_i \subseteq f_i$ *for* $i = 1, \ldots, n$;
3. f_i *is a* χ_i*-permorphism for* $i = 1, \ldots, n$;
4. $\mathrm{fix}(f_1) = \mathrm{fix}(f_2) = \ldots = \mathrm{fix}(f_n) = \{d\}$.

Moreover, $\mathcal{B}$ has the extra properties mentioned in 2.3.4.

Herwig shows that Lemma 2.3.10 follows from Lemma 2.3.11 in the following steps: suppose $\mathcal{A} \in \mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$. First, if $\mathfrak{F}$ is finite, and each χ_i is the identity, it is easy to see that $\mathcal{B}$ is necessarily of link type $\mathfrak{L}$. This is then used to show that if for all $T \in \mathfrak{F}$ every substructure $T' \subseteq T$ is packed, the extension $\mathcal{B}$ of $\mathcal{A}$ given by 2.3.11 is in fact in $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$. In turn, this yields the general case where the only restrictions on $\mathfrak{F}$ are as in the hypothesis of 2.3.10. The permorphisms play no role in Herwig's argument here, therefore his proof goes through to our case.

We shall only sketch Herwig's argument for the proof of Lemma 2.3.11 above. The structure is entirely similar to the argument for K_m-free graphs in the previous section.

Let Δ-types over $\mathcal{A}$ be positive atomic types without equality. A Δ-type is said to be realisable if it does not realise a structure which weakly embeds a structure in $\mathfrak{F}$. The argument then proceeds to show that there is a $\mathfrak{F}$-free extension $\mathcal{C}$ of $\mathcal{A}$ such that the number of realisations in $\mathcal{C}$ of a realisable type p is determined by the size of the set of parameters of p. Moreover, only realisable types are realised in $\mathcal{C}$. The permorphisms p_i are then extended to bijections $h_i \in \mathrm{Sym}(\mathcal{C})$ such that, for all $i \in \{1, \ldots, n\}$ and $b \in \mathcal{C}$

$$\Delta\text{-tp}(b^{h_i}/D_i') = (\Delta\text{-tp}(b/D_i))^{p_i}.$$

This is done pretty much in the same way as in 2.3.7.

The construction of $\mathcal{C}$ is then used to define a set of symbols where arities are reduced by 1: for $R \in S_k$ and $c \in \mathcal{C}$, $R_c^{k'}$ is a $(k-1)$-ary relation symbol interpreted in $\mathcal{A}$ as

$$\mathcal{A} \models R_c^{k'} b_1 \cdots b_{k-1} \iff \mathcal{C} \models R b_1 \cdots b_{k'-1} c b_{k'} \cdots b_{k-1}.$$

Let $S' := S \cup \{R_c^{k'} : k \in \omega, R \in S_k, 1 \leq k' \leq k, c \in \mathcal{C}\}$, and extend χ_i to a permutation χ_i' of S' by $(R_c^{k'})^{\chi_i'} := (R^{\chi_i})^{k'}_{c^{h_i}}$. Clearly, $\mathcal{A}$ can be viewed as an S'-structure.

Then structures in $\mathfrak{F}$ are expanded to S'-structures forming a new finite family $\mathfrak{F}'$ of finite irreflexive packed structures, invariant under the χ_i', and whose maximal size is the maximal size of structures in $\mathfrak{F}$ minus 1. Moreover, $\mathcal{A}$ turns out to be $\mathfrak{F}'$-free, and the p_i are in fact χ_i'-permorphisms. So the inductive hypothesis can be applied to obtain extensions $\mathcal{B}$ of $\mathcal{A}$ and f_i of p_i with the required properties. One can then check that $\mathcal{B}$ is in fact $\mathfrak{F}$-free. Again, the cycle type of the permorphisms involved does not affect this part of the argument, and Herwig's proof goes through unchanged.

The Fraïssé limits of the classes $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$ of structures described in this section have the fixed point extension property and free amalgamation in the sense of Definition 2.1.6. Hence, by Proposition 2.1.9, the conjugacy class on pairs $\mathcal{D}$ (using the notation of 2.1.9) is comeagre in $X_d \times X_d$. Then Lemma 2.2.1 yields the following theorem:

Theorem 2.3.12 *Let $\mathfrak{L}$ be a set of link structures and $\mathfrak{F}$ be a set of finite structures in a finite relational language. Let $\mathcal{M}$ be the Fraïssé limit of the class $\mathfrak{K}_{\mathfrak{L}\mathfrak{F}}$. Then $\mathcal{M}$ has a weak $\forall\exists$ interpretation.*

Chapter 3

Reconstruction of classical geometries

We shall give an application of Rubin's method of weak $\forall\exists$ interpretations to obtain reconstruction results for the projective space $\mathrm{PG}(V)$, where V is a vector space of dimension $\aleph_0$ over a finite field F, and for the projective symplectic, unitary and orthogonal spaces on V. The last section of the chapter contains a reconstruction result for various subgroups of the affine group $\mathrm{AGL}(V)$ acting on V: we show that V, as an affine space, is definable in $\mathrm{AGL}(V)$ and in certain of its subgroups.

In this chapter we work with a slight generalisation of Rubin's definition of weak $\forall\exists$ interpretation:

Definition 3.0.13 *[Generalised weak $\forall\exists$ interpretation] Let $\mathcal{M}$ be ω-categorical. Then $\mathcal{M}$ has a* ***generalised weak $\forall\exists$ interpretation*** *if there are 1-types $P_1, \ldots, P_r$ of $\mathcal{M}$ each of which has a weak $\forall\exists$ interpretation via a conjugacy class on tuples, such that*

1. $\mathcal{M} \subseteq \mathrm{dcl}(\{x : P_1(x) \vee \cdots \vee P_r(x)\})$, *and*

2. $\mathrm{Aut}(\mathcal{M})$ *is faithful and closed in its action on* $\{x : P_1(x) \vee \cdots \vee P_r(x)\}$.

A weak $\forall\exists$ interpretation in the sense of Definition 1.1.6 is a special case of a generalised weak $\forall\exists$ interpretation: just take $\{x : P_1(x) \vee \cdots \vee P_r(x)\} = \mathcal{M}$.

Rubin's reconstruction result holds with this more general definition:

Proposition 3.0.14 *Let $\mathcal{M}$ and $\mathcal{N}$ be ω-categorical and such that* $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$ *as pure groups. Suppose that $\mathcal{M}$ has a generalised weak $\forall\exists$ interpretation. Then $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable.*

Proof Let $P := \{x : P_1(x) \vee \cdots \vee P_r(x)\}$, and suppose

$$\langle \mathrm{Aut}(\mathcal{M}), P\rangle \cong \langle \mathrm{Aut}(\mathcal{M}), \bigcup_{i=1}^{n} C_i/E_i\rangle,$$

where C_i is a conjugacy class on tuples. Since we suppose $\langle \mathrm{Aut}(\mathcal{M}), P\rangle$ to be faithful and closed, an argument entirely similar to 1.1.10 shows that $\langle \mathrm{Aut}(\mathcal{M}), P\rangle$ and $\langle \mathrm{Aut}(\mathcal{N}), \mathcal{N}\rangle$ have the same open subgroups. Since $\mathcal{M} \subseteq \mathrm{dcl}(P)$ by hypothesis, and $\mathrm{dcl}(P)$ is trivially interpretable in $\mathcal{M}$, it is easy to see that $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}\rangle$ has the same open subgroups as $\langle \mathrm{Aut}(\mathcal{M}), P\rangle$. The claim then follows. $\square$

We shall write $\mathrm{GL}(V)$ and $\mathrm{PG}(V)$ for $\mathrm{GL}(\aleph_0, q)$ and $\mathrm{PG}(\aleph_0, q)$ respectively, and similarly for symplectic, unitary and orthogonal groups and their projective versions.

The theorem we prove is the following:

Theorem 3.0.15 *Let V be an $\aleph_0$-dimensional vector space over a finite field F_q, and let $\mathcal{M}$ be an ω-categorical structure with domain* $\mathrm{PG}(V)$ *and such that one of the following holds*[1]*:*

1. $\mathrm{PGL}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma L}(V)$
2. $\mathrm{PSp}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma Sp}(V)$
3. $\mathrm{PU}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma U}(V)$
4. $\mathrm{PO}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma O}(V)$

[1]see Section 3.3 for definitions of the groups $\mathrm{P\Gamma L}(V)$, $\mathrm{P\Gamma Sp}(V)$, $\mathrm{P\Gamma U}(V)$ and $\mathrm{P\Gamma O}(V)$

Then $\mathcal{M}$ has a generalised weak $\forall\exists$ interpretation.

The proof is contained in 1.5.1, 1.5.2, 3.2.10, 3.2.11, 3.3.13, 3.3.14, 3.3.24, 3.3.29.

Remark 3.0.16 The case of a vector space over a field of even characteristic which is smoothly approximated by a sequence of odd dimensional orthogonal geometries is not covered by the above theorem, as the geometry has non trivial radical. However, the abstract group is isomorphic to a symplectic group, so the reconstruction problem is solved by case 2. above.

It should be mentioned that reconstruction results were already known for the above permutation groups, since they have the small index property [11]. What is new is that these structures have weak $\forall\exists$ interpretations, and it may be new even that they are parameter-interpretable in their automorphism groups.

3.1 Preliminaries

If V is a countably infinite dimensional vector space over a finite field F, V is determined up to isomorphism by its dimension, so it is an ω-categorical structure, and so are the symplectic, unitary and orthogonal spaces (V, β, Q) (where β is a sesquilinear form and Q the associated quadratic form in the orthogonal case). The projective spaces corresponding to these spaces are also ω-categorical. We shall produce weak $\forall\exists$ interpretations for various groups acting on $\mathrm{PG}(V)$ and on projective spaces with forms. We concentrate on the reconstruction of the projective spaces, rather than the vector space itself, because reconstruction for V via a weak $\forall\exists$ interpretation cannot be obtained in general, as lemma 3.1.1 will show. Below we take $\mathrm{Aut}(V)$ to be the general linear group $\mathrm{GL}(V)$.

Lemma 3.1.1 *Let V be as above, and suppose $F \neq \mathbb{F}_2$. Then there is no weak $\forall\exists$ interpretation for $\langle \mathrm{GL}(V), V \rangle$.*

Proof The centre $Z(\mathrm{GL}(V))$ is nontrivial, since $Z(\mathrm{GL}(V)) = \{\alpha \mathrm{id}_{\mathrm{GL}(V)} : \alpha \in F \setminus \{0\}\}$. By 1.2.1, the claim follows. $\square$

The proof of Lemma 3.1.1 suggests that the problem with a weak $\forall\exists$ interpretation for $\langle GL(V), V\rangle$ is created by scalars, so it is natural to turn our attention to the projective space $\mathrm{PG}(V)$, whose domain is the set of one-dimensional subspaces of V. There are various closed groups acting on $PG(V)$. The most natural group to consider is $\mathrm{PGL}(V)$: when $\dim(V) = \aleph_0$, $\mathrm{PGL}(V)$ is simple. We have $\mathrm{PGL}(V) \trianglelefteq \mathrm{P\Gamma L}(V)$, where $\mathrm{P\Gamma L}(V)$ is the group of projective semilinear transformations on V, defined as

$$\mathrm{P\Gamma L}(V) := \Gamma\mathrm{L}(V)/\{\alpha \mathrm{id}_{\mathrm{GL}(V)} : \alpha \in F \setminus \{0\}\},$$

where $\Gamma\mathrm{L}(V)$ is the group of all semilinear transformations of V, i.e. maps $f : V \to V$ such that, for some $\sigma \in \mathrm{Aut}(F)$

$$(av + bw)f = a^\sigma(vf) + b^\sigma(wf)$$

for all $a, b \in F$ and $v, w \in V$. The group $\mathrm{P\Gamma L}(V)$ is closed and hence the automorphism group of a structure with domain $\mathrm{PG}(V)$.

Recall that if $G \leq \mathrm{Sym}(\Omega)$ is a closed subgroup of the full symmetric group of a countable set Ω, we can impose a *canonical structure* $\mathcal{O}$ on Ω in a canonical language L, where L contains an n-ary relation symbol R_Δ for each orbit Δ of G on Ω^n. If G acts oligomorphically on Ω, as in our case, the canonical structure is the structure on Ω whose 0-definable relations are determined by the action of the automorphism group. Any structure on Ω with G as automorphism group has the same 0-definable relations as the canonical structure. We shall henceforth assume that the structures we are working with are the canonical ones, determined by the action of the groups considered, and thus we shall not specify the language.

Our aim is to obtain a weak $\forall\exists$ interpretation for all the structures living on $\mathrm{PG}(V)$ determined by those closed groups H acting on $\mathrm{PG}(V)$ such that $\mathrm{PGL}(V) \leq H \leq \mathrm{P\Gamma L}(V)$. These include $\mathrm{PGL}(V)$, $\mathrm{P\Gamma L}(V)$ and some intermediate subgroups of $\mathrm{P\Gamma L}(V)$ which form a normal series. Proposition 1.5.1 will ensure that it suffices to find a weak $\forall\exists$ interpretation for the structure determined by

$\mathrm{PGL}(V)$ in order to have interpretations for the whole range of structures between $\langle \mathrm{PGL}(V), \mathrm{PG}(V)\rangle$ and $\langle \mathrm{P\Gamma L}(V), \mathrm{PG}(V)\rangle$.

Our result on $\mathrm{PG}(V)$ rests on the definition of weak $\forall\exists$ interpretation generalized to conjugacy classes on tuples, mentioned in remark 3.0.13. Our weak $\forall\exists$ interpretation will be based on a conjugacy class on pairs of automorphisms, where elements in each pair share exactly one fixed point. Moreover we shall work with various closed subgroups of $\mathrm{Sym}(\mathrm{PG}(V))$, and for this we shall use Proposition 1.5.1 and Lemma 3.2.12 below. When treating spaces with forms, we shall use 1.5.1 with $n = 1$.

3.2 A weak $\forall\exists$ interpretation for $\mathrm{PG}(V)$

We shall define a conjugacy class on pairs $C \subset \mathrm{PGL}(V) \times \mathrm{PGL}(V)$ and an equivalence relation E on C, $\exists$ definable in the language of groups with parameters such that

$$\langle \mathrm{PGL}(V), \mathrm{PG}(V)\rangle \cong \langle \mathrm{PGL}(V), C/E\rangle$$

as permutation groups. Given the extension of Rubin's definition in [25] to weak $\forall\exists$ interpretations defined with a conjugacy class on a tuple, and Lemma 1.1.3, we shall obtain a weak $\forall\exists$ interpretation for $\mathrm{PGL}(V)$ acting on $\mathrm{PG}(V)$. By Lemma 3.2.12 below, a weak $\forall\exists$ interpretation for $\langle \mathrm{PGL}(V), \mathrm{PG}(V)\rangle$ suffices to cover all the structures on $\mathrm{PG}(V)$ determined by those closed groups H such that $\mathrm{PGL}(V) \trianglelefteq H \trianglelefteq \mathrm{P\Gamma L}(V)$. Since $\mathrm{P\Gamma L}(V)/\mathrm{PGL}(V)$ is a finite cyclic group, these groups are closely related.

3.2.1 Transvections

Definition 3.2.1 *A* ***transvection*** *is $\tau \in \mathrm{GL}(V)$ such that there are a linear functional $u(x)$ in the dual space V' and a vector $d \in V \setminus \{0\}$ such that*

- $d\tau = d$

- $x\tau = x + u(x)d$ *for all* $x \in V$

We shall write $\tau_{d,u}$ for the transvection above. We shall call $\langle d \rangle$ the **direction** of τ.

The linear functional $u(x)$ will define a hyperplane U of equation $u(x) = 0$ and τ fixes U pointwise. Also, $d\tau = d$, hence $d \in U$. Given a transvection τ, we shall indicate the direction of τ by d_τ, and the fixed hyperplane by U_τ. Different transvections might have the same direction and the same fixed hyperplane:

Proposition 3.2.2 *Let λ be a scalar, u, u' non zerolinear functionals and d, d' nonzero vectors. Then*

1. $\tau_{\lambda d,u} = \tau_{d,\lambda u}$
2. $\tau_{d,u} = \tau_{d',u'}$ *if and only if there is a nonzero scalar μ such that $d' = \mu d$ and $u' = \mu^{-1}u$.*

Proof By direct calculation, using the formula defining a transvection. □

We also recall the following facts about transvections ([24] and [5] prove these properties for a finite-dimensional V, but the arguments carry through to the $\aleph_0$-dimensional case): [5] and [24]):

Lemma 3.2.3 *If $g \in \mathrm{GL}(V)$ and $\tau_{d,u} \in T$, $\tau_{d,u}^g = \tau_{d^g, g^{-1}u}$.*

[5] 2.4.3. □

Proposition 3.2.4 *There is a conjugacy class T in $\mathrm{GL}(V)$ consisting of all the transvections.*

Proof [5] 2.4.4. □

Proposition 3.2.5 *Let τ and σ be nontrivial transvections in $\mathrm{GL}(V)$. Then $\tau\sigma$ is a transvection if and only if $U_\tau = U_\sigma$ or $\langle d_\tau \rangle = \langle d_\sigma \rangle$.*

Proof [24] 1.17. □

Lemma 3.2.6 *The mapping* $\widehat{\ }: \mathrm{GL}(V) \to \mathrm{PGL}(V)$ *which takes* $g \in \mathrm{GL}(V)$ *to the mapping* $\hat{g}$ *defined by*

$$\hat{g}(\langle v \rangle) := \langle g(v) \rangle$$

is a group homomorphism which is continuous and open.

Proof It is easy to check that $\widehat{\ }$ is a group homomorphism. To prove that $\widehat{\ }$ is continuous, consider a basic open set in $\mathrm{PGL}(V)$, say

$$\hat{U} = \{\hat{g} \in \mathrm{PGL}(V) : \langle v_i \rangle^{\hat{g}} = \langle w_i \rangle, i = 1, \ldots, n\}.$$

The inverse image of $\hat{U}$ is

$$U = \bigcup_{\bar{\alpha},\bar{\beta} \in (F \setminus \{0\})^n} \{g \in \mathrm{GL}(V) : \alpha_i v_i^g = \beta_i w_i, i = 1, \ldots n\}$$

which is a union of open sets, hence it is open.

Clearly if $g \in \mathrm{GL}(V)$ is such that $v^g = v$, then $\langle v \rangle^{\hat{g}} = \langle v \rangle$, so $\widehat{\mathrm{Stab}_{\mathrm{GL}(V)}}(v_1 \ldots v_n) \supseteq \mathrm{Stab}_{\mathrm{PGL}(V)}(\langle v_1 \rangle \ldots \langle v_n \rangle)$. Hence the image of an open set is again open. $\square$

Definition 3.2.7 *We define* $\hat{\tau} \in \mathrm{PGL}(V)$ *to be a* ***projective transvection*** *if it is the image under* $\widehat{\ }$ *of some transvection* $\tau \in \mathrm{GL}(V)$*.*

Since $\widehat{\ }$ is a homomorphism, by 3.2.4 projective transvections form a complete conjugacy class $\widehat{T}$ in $\mathrm{PGL}(V)$.

Lemma 3.2.8

1. *The preimage of the projective transvection* $\hat{\tau}$ *under* $\widehat{\ }$ *contains all nonzero scalar multiples of* τ *and nothing else.*
2. *A scalar multiple of a transvection in general is not a transvection. In particular, if* τ, σ *are transvections then* $\lambda\tau = \sigma \iff \lambda = 1$ *and* $\tau = \sigma$*.*

Proof [24] 1.15. $\square$

It follows that $\hat{\tau} \in \mathrm{PGL}(V)$ is a projective transvection if and only if there is $\tau \in T$ whose image under $\widehat{\ }$ is $\hat{\tau}$. Such a τ is unique and we shall call it the transvection **associated with** $\hat{\tau}$. Hence to each projective transvection $\hat{\tau}$ there remain associated a unique fixed hyperplane $U_{\hat{\tau}}$ and a unique direction $\langle d_{\hat{\tau}} \rangle$, which are those of the associated transvection. In what follows we shall always assume that τ is the transvection associated with $\hat{\tau}$ and that U_τ, $\langle d_\tau \rangle$ are the corresponding hyperplane and direction. From these considerations it is easy to obtain a projective version of Proposition 3.2.5:

Proposition 3.2.9 *Let $\hat{\tau}, \hat{\sigma}$ be non trivial projective transvections. Then $\hat{\tau}\hat{\sigma}$ is a transvection if and only if $U_\tau = U_\sigma$ or $\langle d_\tau \rangle = \langle d_\sigma \rangle$.*

Proof [24], 1.23 $\square$

3.2.2 The interpretation

We shall select a conjugacy class of pairs of projective transvections $\hat{C} \subseteq \mathrm{PGL}(V) \times \mathrm{PGL}(V)$ so that transvections in the same pair have the same direction and different fixed hyperplane, and an equivalence relation E on $\hat{C}$ identifying pairs having the same direction.

Proposition 3.2.10 *Let $(\hat{\sigma}, \hat{\sigma}') \in \mathrm{PGL}(V) \times \mathrm{PGL}(V)$ be a pair of transvections such that $\langle d_\sigma \rangle = \langle d_{\sigma'} \rangle$ and $U_\sigma \neq U_{\sigma'}$. Then for all $\langle d \rangle \in \mathrm{PG}(V)$ there are $\hat{g} \in \mathrm{PGL}(V)$ and a pair $(\hat{\tau}, \hat{\tau}')$ of transvections such that $(\hat{\sigma}, \hat{\sigma}')^{\hat{g}} = (\hat{\tau}, \hat{\tau}')$ and $\langle d \rangle = \langle d_\tau \rangle = \langle d_{\tau'} \rangle$, $U_\tau \neq U_{\tau'}$.*

Proof Clearly, if $(\hat{\sigma}, \hat{\sigma}')$ is such that $\langle d_\sigma \rangle = \langle d_{\sigma'} \rangle$ and $U_\sigma \neq U_{\sigma'}$, and $\hat{g} \in \mathrm{PGL}(V)$ is such that $\hat{\sigma}^{\hat{g}} = \hat{\tau}$, $(\sigma')^g = \tau'$, then (by 3.2.3) $\langle d_\tau \rangle = \langle d_{\tau'} \rangle$ and $U_\tau \neq U_{\tau'}$. Since $\mathrm{GL}(V)$ is transitive on the points of $\mathrm{PG}(V)$, given any $\langle d \rangle \in \mathrm{PG}(V)$ we can find $g \in \mathrm{GL}(V)$, hence $\hat{g} \in \mathrm{PGL}(V)$, such that $\langle d_\sigma \rangle^{\hat{g}} = \langle d \rangle$. Then $(\hat{\sigma}, \hat{\sigma}')^{\hat{g}}$ will be our required pair. $\square$

This lemma ensures that all points of $\mathrm{PG}(V)$ are represented by at least a pair in $\hat{C} = \{(\hat{\sigma}, \hat{\sigma}')^{\hat{g}} : \hat{g} \in \mathrm{PGL}(V)\}$. We can now obtain an $\forall\exists$ formula in the language of groups which defines pairs of transvections representing the same point of $\mathrm{PG}(V)$.

Proposition 3.2.11 *Let* $(\hat{\rho}, \hat{\rho}')$ *and* $(\hat{\sigma}, \hat{\sigma}')$ *be in* $\hat{C}$ *as defined above. Then*

$$(\hat{\rho}, \hat{\rho}')E(\hat{\sigma}, \hat{\sigma}') \textit{ iff } \langle d_\rho \rangle = \langle d_\sigma \rangle$$

is a conjugacy invariant equivalence relation on $\hat{C}$*,* $\exists$ *definable in* $\mathrm{PGL}(V)$ *with parameters in the language of groups. Hence there is an* $\forall\exists$ *equivalence formula* ϕ *in the language of groups defining* E*.*

Proof Suppose $(\hat{\sigma}, \hat{\sigma}')$ is in $\hat{C}$ (so $\langle d_\sigma \rangle = \langle d_{\sigma'} \rangle$ and $U_\sigma \neq U_{\sigma'}$). We claim that $\langle d_\sigma \rangle = \langle d_\rho \rangle$ if and only if the products $\hat{\sigma}\hat{\rho}$ and $\hat{\sigma}\hat{\rho}'$ are both projective transvections.

By 3.2.9, $\langle d_\sigma \rangle = \langle d_\rho \rangle$ implies that $\hat{\sigma}\hat{\rho}$ and $\hat{\sigma}\hat{\rho}'$ are projective transvections. To prove the converse, suppose for a contradiction that $\hat{\sigma}\hat{\rho}$ and $\hat{\sigma}\hat{\rho}'$ are projective transvections, yet $\langle d_\sigma \rangle \neq \langle d_\rho \rangle$ (hence also $\langle d_\sigma \rangle \neq \langle d_{\rho'} \rangle$). Then by 3.2.9 $\hat{\sigma}\hat{\rho}$ is a transvection if and only if $U_\sigma = U_\rho$. Likewise, $\hat{\sigma}\hat{\rho}'$ is a transvection if and only if $U_\sigma = U_{\rho'}$. But then $U_\rho = U_{\rho'}$, contradicting $(\rho, \rho') \in C$.

Hence the formula

$\phi(x, x', y, y') \equiv xy$ is a projective transvection and xy' is a projective transvection

defines the equivalence relation E in the language of groups. By 3.2.3 E is conjugacy invariant. Note that the property of being a projective transvection is definable with a single parameter (say $\hat{\sigma}$) by the existence of a conjugating element to $\hat{\sigma}$ (by 3.2.4), so ϕ is in fact

$$\exists w \exists z \; ((xy)^w = \hat{\sigma} \wedge (xy')^z = \hat{\sigma}),$$

which is an existential formula. By Lemma 1.1.3, the formula

$$\phi(\hat{\sigma}, x, y) \wedge \text{`}\phi(\hat{\sigma}, x, y) \text{ defines an equivalence relation on } C\text{'},$$

where C is the conjugacy class of the parameter $\hat{\sigma}$, is an $\forall\exists$ equivalence formula. $\square$

The following Lemma allows to lift our weak $\forall\exists$ interpretation for $\langle \mathrm{PGL}(V), \mathrm{PG}(V) \rangle$ to larger subgroups of $\mathrm{P\Gamma L}(V)$. The same role will be played by Proposition 1.5.1 for spaces with forms.

Lemma 3.2.12 *Let G be a closed group and such that $\mathrm{PGL}(V) \leq G \leq \mathrm{P\Gamma L}(V)$. Then $\langle G, \mathrm{PG}(V)\rangle$ has a weak $\forall\exists$ interpretation.*

Proof Let $\hat{C} = (\hat{\sigma}, \hat{\sigma}')^{\mathrm{PGL}(V)}$ be the conjugacy class on pairs of transvections which gives the weak $\forall\exists$ interpretation of Proposition 3.2.11 above.
Since $\mathrm{PGL}(V) \lhd G$, we have $\hat{C} \leq (\hat{\sigma}, \hat{\sigma}')^G \subseteq \mathrm{PGL}(V)$. Hence $(\hat{\sigma}, \hat{\sigma}')^G$ is again made of pairs of transvections $(\hat{\rho}, \hat{\rho}')$ such that $\langle d_\rho\rangle = \langle d_{\rho'}\rangle$ but $U_\rho = U_{\rho'}$. Then we define $\hat{E}$ on $(\hat{\sigma}, \hat{\sigma}')^G$ with exactly the same formula as in 3.2.11, so that $(\hat{\rho}, \hat{\rho}')\hat{E}(\hat{\sigma}, \hat{\sigma}')$ iff $\langle d_\rho\rangle = \langle d_\sigma\rangle$. $\square$

3.3 Spaces with forms

Let V be a vector space as above, and suppose $\sigma \in \mathrm{Aut}(F)$. Let us recall some basic definitions and notation. A **sesquilinear form** on V is a map $\beta : V \times V \to F$ such that for all $u_i, v_i \in V$, $a, b \in F$

1. $\beta(u_1 + u_2, v) = \beta(u_1, v) + \beta(u_2, v)$
2. $\beta(u, v_1 + v_2) = \beta(u, v_1) + \beta(u, v_2)$
3. $\beta(au, bv) = ab^\sigma \beta(u, v)$

The form β is said to be

- **alternating** if $\sigma = 1 \in \mathrm{Aut}(F)$ and $\beta(v, v) = 0$ for all v in V;
- **symmetric** if $\sigma = 1 \in \mathrm{Aut}(F)$ and $\beta(u, v) = \beta(v, u)$ for all u, v in V;
- **hermitian** if $\sigma \neq 1, \sigma^2 = 1 \in \mathrm{Aut}(F)$ and $\beta(u, v) = \beta(v, u)^\sigma$ for all u, v in V.

If β is alternating then $\beta(u, v) = -\beta(v, u)$ for all $u, v \in V$.

If X is a subspace of V we define $X^\perp := \{u \in V : \forall x \in X\ \beta(u,x) = 0\}$. Note that $X^\perp \leq V$. The **radical** of V is $V^\perp$. If $U \leq V$, $\mathrm{Rad}(U) = U \cap U^\perp$. The form β is said to be **nondegenerate** if $\mathrm{Rad}(V) = \{0\}$.

A **quadratic form** on V is a function $Q : V \to F$ such that

$$Q(av) = a^2 Q(v) \text{ for all } a \in F, v \in V, \text{ and}$$

$$\beta(u,v) := Q(u+v) - Q(u) - Q(v)$$

is a bilinear form (i.e. sesquilinear with $\sigma = 1$). Then β is symmetric, and it is called the bilinear form **associated with** Q. An easy calculation shows that Q determines β and $\beta(u,u) = 2Q(u)$. If $\mathrm{char}(F) = 2$, we get that $\beta(u,u) = 0$ for all $u \in V$.

The forms defined above give rise to three kinds of spaces:

- the symplectic space (V, β), where β is alternating nondegenerate;
- the orthogonal space (V, β, Q), where β is symmetric nondegenerate;
- the unitary space (V, β), where β is hermitian nondegenerate.

If V is countably infinite dimensional and F is finite then each form is unique up to isomorphism, so the space (V, β) is an ω-categorical structure. Unlike the vector space case, categoricity does not hold in uncountable dimension. Our convention about adopting the canonical language will hold for spaces with forms.

Definition 3.3.1 *If (V_1, β_1, Q_1) and (V_2, β_2, Q_2) are F vector spaces as above (both symplectic or both orthogonal or both unitary) then $f : V_1 \to V_2$ is a* ***linear isometry*** *if f is linear and for all $u, v \in V_1$*

$$\beta_2(uf, vf) = \beta_1(u,v) \text{ and } Q_2(vf) = Q_1(v).$$

We shall denote the isometry group of the space (V, β, Q) by $\mathrm{O}(V, \beta, Q)$. This notation covers the symmetric, the unitary and the orthogonal groups. We shall write $\mathrm{Sp}(V)$ and $\mathrm{PSp}(V)$ for the symplectic and projective symplectic groups respectively. $\mathrm{O}(V)$, $\mathrm{PO}(V)$, $\mathrm{U}(V)$ and $\mathrm{PU}(V)$ will denote the orthogonal, projective orthogonal, unitary and projective unitary groups respectively. We also need to define

- $\Gamma\mathrm{Sp}(V) := \{f \in \Gamma\mathrm{L}(V) : f$ is τ is -semilinear for some $\tau \in \mathrm{Aut}(F)$ and $\exists a \in F : \forall u, v \in V\ \beta(uf, vf) = a(\beta(u,v)^{\tau})\}$;
- $\Gamma\mathrm{U}(V) := \{f \in \Gamma\mathrm{L}(V) : f$ τ-semilinear for some $\tau \in \mathrm{Aut}(F)$ and $\exists a \in F : a^{\sigma} = a$ and $\forall u, v \in V\ \beta(uf, vf) = a(\beta(u,v)^{\tau})\}$;
- $\Gamma\mathrm{O}(V) := \{f \in \Gamma\mathrm{L}(V) : f$ is τ-semilinear for some $\tau \in \mathrm{Aut}(F)$ and $\exists a \in F : \forall v \in V\ Q(vf) = aQ(v)^{\tau}\}$.

The projective versions $\mathrm{P\Gamma Sp}(V)$, $\mathrm{P\Gamma U}(V)$ and $\mathrm{P\Gamma O}(V)$ of these groups are obtained in the usual way by quotienting $\Gamma\mathrm{Sp}(V)$, $\Gamma\mathrm{U}(V)$ and $\Gamma\mathrm{O}(V)$ by scalars. We now need more definitions:

Definition 3.3.2 *1. A non-zero vector $v \in V$ is **isotropic** if $\beta(v,v) = 0$;*

*2. a subspace $W \subseteq V$ is **totally isotropic** if $W \subseteq W^{\perp}$;*

*3. a non-zero vector v is **singular** if $Q(v) = 0$ (note that in odd characteristic a vector is singular if and only if it is isotropic);*

*4. $W \subseteq V$ is **totally singular** if $Q(w) = 0$ for all $w \in W$;*

*5. $W \subseteq V$ is **non-degenerate** if $W \cap W^{\perp} = \{0\}$;*

*6. if $W = U \oplus V$ and $\beta(u,v) = 0$ for all $u \in U$, $v \in V$, we say that W is an **orthogonal direct sum** of U and V, and we write $U \perp V$.*

Definition 3.3.3 *A pair of vectors u, v such that u, v are both isotropic and $\beta(u,v) = 1$ is called a **hyperbolic pair**, and the line $\langle u, v\rangle$ in $\mathrm{PG}(V)$ is a **hyperbolic line**. In the presence of a quadratic form Q, we also require that $Q(u) = Q(v) = 0$.*

We shall now state Witt's theorem, which is a major result concerning spaces with forms and which we shall repeatedly need:

Theorem 3.3.4 (Witt) *Let V be a symplectic, orthogonal or unitary space, where V has dimension $\aleph_0$ over the finite field F, and let $U \leq V$ be a finite dimensional subspace. Suppose that $g : U \to V$ is a linear isometry. Then the following are equivalent:*

1. *there is a linear isometry $h : V \to V$ such that $ug = uh$ for all $u \in U$;*

2. *$(U \cap \mathrm{Rad}(V))g = Ug \cap \mathrm{Rad}(V)$.*

Proof [27], 7.4. $\square$

In particular any isomorphism between two nondegenerate subspaces of V can be extended to a full isomorphism.

3.3.1 Generics in $\mathrm{PO}(V, \beta, Q)$

In this section we shall establish some facts about isometry groups which are needed later for finding weak $\forall\exists$ interpretations for spaces with forms.

We shall think of (V, β, Q) as the Fraïssé limit of finite dimensional spaces having a hyperbolic basis. We refer the reader to the literature for proofs that, when the underlying field is finite, the even dimensional vector space U can be equipped with an orthogonal and unitary form admitting a hyperbolic basis. We shall show that $\mathrm{O}(V, \beta, Q)$ contains a generic automorphism, so that 1.5.1 applies (by 1.5.2).

Lemma 3.3.5 *Let $\mathcal{C}$ be the class of finite-dimensional non degenerate spaces $\langle (U, \beta, Q), f \rangle$ over a given finite field F having a hyperbolic basis, with $f \in \mathrm{O}(U, \beta, Q)$. Then $\mathcal{C}$ has a Fraïssé limit $\langle (V, \beta, Q), f \rangle$ such that:*

1. *the class of finite-dimensional non degenerate subspaces of $\langle (V, \beta, Q), f \rangle$ is equal to $\mathcal{C}$;*

2. *$\langle (V, \beta, Q), f \rangle$ is a union of a chain of finite-dimensional nondegenerate subspaces;*

3. *if* $\langle (U, \beta, Q), f\rangle$ *is a finite-dimensional nondegenerate subspace of* $\langle (V, \beta, Q), f\rangle$ *and* $\alpha : \langle (U, \beta, Q), f\rangle \to \langle (W, \beta, Q), f\rangle$ *is an embedding whose range is nondegenerate, then there is a subspace* $\langle (W, \beta, Q), f\rangle$ *embeds in* $\langle (V, \beta, Q), f\rangle$ *over* U.

Moreover, $\langle (V, \beta, Q), f\rangle$ *is unique among countable structures satisfying properties 1., 2. and 3., and any isomorphism between nondegenerate finite-dimensional subspaces of* $\langle (V, \beta, Q), f\rangle$ *extends to an automorphism of* $\langle (V, \beta, Q), f\rangle$.

Proof Following Evans ([12], p. 44), we say that a class of $\mathcal{C}$-*embeddings* is a collection $\mathcal{E}$ of embeddings $\alpha : A \to B$ with A, $B \in \mathcal{C}$ such that:

1. isomorphisms are in $\mathcal{E}$;
2. $\mathcal{E}$ is closed under composition;
3. if $\alpha : A \to B$ is in $\mathcal{E}$ and $C \subseteq B$ is a substructure in $\mathcal{C}$ such that $\alpha(A) \subseteq C$, then the map obtained by restricting the range of α to C is also in $\mathcal{E}$.

It is clear that if we take the class of embeddings whose range is a structure in $\mathcal{C}$, i.e. whose range is nondegenerate, we obtain a class of $\mathcal{C}$-embeddings. It suffices to prove that $\mathcal{E}$ satisfies

AP' If $\langle U, h\rangle$, $\langle V_1, g_1\rangle$, $\langle V_2, g_2\rangle$ are in $\mathcal{C}$ and $\alpha_i : \langle U, h\rangle \to \langle V_i, g_i\rangle$, $i = 1, 2$, are embeddings in $\mathcal{E}$, then there are $\langle W, g\rangle \in \mathcal{C}$ and $\gamma_i : \langle V_i, g_i\rangle \to \langle W, g\rangle$, $\gamma_i \in \mathcal{E}$, such that $\alpha_1\gamma_1 = \alpha_2\gamma_2$;

JEP' if $\langle V_1, g_1\rangle$ and $\langle V_2, g_2\rangle$ are in $\mathcal{C}$, there is $\langle W, g\rangle \in \mathcal{C}$ and embeddings $\alpha_i : V_i \to W$ such that $\alpha_i \in \mathcal{E}$ is in $\mathcal{C}$.

We prove the amalgamation property AP'. By identifying U with $\alpha_i(U)$ we may assume that $\alpha_1 = \mathrm{id}$ and $\alpha_2 = \mathrm{id}$, so that $\langle U, h\rangle \subseteq \langle V_i, f_1\rangle$, $i = 1, 2$. Choose a hyperbolic basis $\mathcal{B} = \{e_1, f_1, \ldots, e_n, f_n\}$ for U, where each pair (e_i, f_i) is hyperbolic. Extend $\mathcal{B}$ to hyperbolic bases $\mathcal{B}_1 = \mathcal{B} \cup \{e_{n+1}, f_{n+1}, \ldots, e_r, f_r\}$ for

V_1 and $\mathcal{B}_2 = \mathcal{B} \cup \{e'_{n+1}, f'_{n+1}, \ldots, e'_s, f'_s\}$ for V_2. Let $W = \langle \mathcal{B}_1 \cup \mathcal{B}_2 \rangle$, $g = g_1 \cup g_2$ and define $\beta(e_i, e'_j) = \beta(f_i, f'_j) = \beta(e_i, f'_j) = \beta(e'_j, f_i) = 0$ for $i = n+1, \ldots, r$ and $j = n+1, \ldots, s$. It is easy to check that g respects β on this basis, hence $g \in O(W, \beta, Q)$ is the required extension of f.

The joint embedding property JEP' is proved similarly. □

We prove that f (as obtained in the last lemma) is generic, using Banach Mazur games (cf. [21]. 8.H). Lemma 3.3.7 below is well known, and it is central to the fact that vector spaces over finite fields with nondegenerate bilinear forms are smoothly approximable (cf. [8]).

Lemma 3.3.6 *Let $u \in (V, \beta, Q)$, $u \neq 0$ and isotropic, and let $w \in V \setminus u^{\perp}$. Then $L = \langle u, w \rangle$ is a hyperbolic plane and u is contained in a hyperbolic pair of L.*

Proof [2], 19.12. □

Lemma 3.3.7 *Let V be a countably infinite dimensional vector space over a finite field F with a nondegenerate form β, and let $U \leq V$ be a finite dimensional subspace. Then there is a finite dimensional $\bar{U} \leq V$ such that $\bar{U}$ is nondegenerate and $U \leq \bar{U}$.*

Proof Recall that $\mathrm{Rad}(U) = U^{\perp} \cap U$. Since U is finite dimensional, $U = (U^{\perp})^{\perp}$. If U is degenerate, then $\mathrm{Rad}(U) \neq \{0\}$, and we can write $U = \mathrm{Rad}(U) \perp W$, with W nondegenerate. Let $u_1, \ldots, u_r$ be a basis for $\mathrm{Rad}(U)$. We claim that we can find $v_1, \ldots, v_r \in V$ such that, for $i = 1, \ldots, r$, $P_i = \langle u_i, v_i \rangle$ is a hyperbolic pair and P_i is orthogonal to $\langle u_1, \ldots, u_{i-1} \rangle \perp W$. Assume inductively that the claim holds for $U_0 = \langle u_1, \ldots, u_{r-1} \rangle \perp W$. We have that

$$\begin{aligned}
\mathrm{Rad}(U_0^{\perp}) &= U_0^{\perp} \cap (U_0^{\perp})^{\perp} \\
&= U_0^{\perp} \cap U_0 \\
&= \mathrm{Rad}(U_0) \\
&= \langle u_1, \ldots, u_{r-1} \rangle
\end{aligned}$$

So $u_r \in U_0^\perp$, yet $u_r \notin \mathrm{Rad}(U_0^\perp)$. Therefore there must be a vector $v_r \in U_0^\perp$ such that $\beta(u_r, v_r) \neq 0$. By Lemma 3.3.6 we may assume that (u_r, v_r) is a hyperbolic pair, and we let $P_r = \langle u_r, v_r \rangle$. By the inductive hypothesis, there are $P_1, \ldots, P_{r-1}$ which are mutually orthogonal and orthogonal to W. Let $\bar{U} := P_1 \perp P_2 \perp \ldots \perp P_r \perp W$. Then $\bar{U}$ is nondegenerate, finite dimensional and contains U, as required. $\square$

Proposition 3.3.8 *The isometry f built above is a generic.*

Proof Let C be the conjugacy class of f. We show C is comeagre by playing the Banach-Mazur game of C.

Let $\mathbb{P} = \{g : V \to V \text{ s.t. } g \text{ is a partial finite isometry}\}$. Note that $\mathbb{P}$ is partially ordered by inclusion. The game is played as follows: players I and II choose an increasing sequence of elements of $\mathbb{P}$

$$p_1 \leq p_2 \leq \ldots \leq p_n \leq p_{n+1} \leq \ldots$$

Player I starts the game and chooses p_i for i odd, player II chooses at even stages. Player II wins if and only if $p := \bigcup_{i \in \omega} p_i \in C$. Player II has a winning strategy iff C is comeagre in $O(V, \beta, Q)$.
Enumerate V. Player II can always play so that at stage i, for $i > 1$ and even:

1. $v_i \in \mathrm{dom}(p_i)$;

2. $\mathrm{dom}(p_i) = \mathrm{ran}(p_i)$;

3. $\langle V, p \rangle$ is weakly homogeneous, that is: if $\langle A, p_A \rangle$ and $\langle B, p_B \rangle$ are finitely generated substructures of $\langle V, f \rangle$ such that A, B are nondegenerate, $\langle A, p_A \rangle \subseteq \langle B, p_B \rangle$ and $\phi : \langle A, p_A \rangle \to \langle V, p \rangle$ is an embedding, then there is an embedding $\hat{\phi} : \langle B, p_B \rangle \to \langle V, p \rangle$ extending ϕ.

Let $U_i = \mathrm{dom}(p_i)$ and $V_i = \mathrm{ran}(p_i)$. Points 1. and 2. are a consequence of 3.3.7, for: given any $v \in V$, player II can embed the space $\langle U_i, V_i, v \rangle$ generated by

U_i, V_i and v in a nondegenerate finite dimensional space U_{i+1} and then use Witt's theorem to extend p_i to p_{i+1} defined on U_{i+1}.

Player II must also ensure that 3. holds. He can achieve this if he ensures that: given $\langle U_{i-1}, p_{i-1}\rangle$, and $\langle A, p_A\rangle \to \langle U_{i-1}, p_{i-1}\rangle$ and $\langle B, p_b\rangle \supseteq \langle A, p_A\rangle$, where A, B are nondegenerate, then $\langle B, p_B\rangle$ can be embedded in $\langle U_i, p_i\rangle$. So II really wants to create an amalgam of U_{i-1} and B over A. First, II extends U_{i-1} to $\widehat{U}_{i-1}$ nondegenerate (by 3.3.7). He also ensures that p_{i-1} extends to an automorphism $\hat{p}_{i-1}$ of $\widehat{U}_{i-1}$. Then he amalgamates $\widehat{U}_{i-1}$ and B over A as in 3.3.5. Call this amalgam U'_i. By universality of (V, β, Q) as a space with a form (which follows from our construction of V as a Fraïssé limit), II can find a copy of U'_i and by homogeneity he can choose it to be over $\widehat{U}_{i-1}$, say $\psi : U'_i \to V$ with $U_{i-1} \subseteq \psi(U'_i)$. So II can put $U_i := \psi(U'_i)$ and define $p_i := p'_i \circ \psi$. □

Proposition 3.3.9 *Let g be a generic automorphism of $G \leq \mathrm{O}(V, \beta, Q)$. Then the projective image $\hat{g}$ is generic in $\hat{G}$.*

Proof Let $C = g^G$ be the comeagre conjugacy class of g. Let $Z = Z(G) \leq \{\alpha I : \alpha \in F, I = \mathrm{id}_G\}$ be the centre of G.

First, we claim that for any $z \in Z$, $zC = C$. Indeed, clearly zC is comeagre (as translation by z is a homeomorphism of G), and $zC = \{zf^{-1}gf : f \in G\} = \{f^{-1}zgf : f \in G\} = (zg)^G$. Hence zC is a conjugacy class. As there is a unique comeagre conjugacy class, $zC = C$. Hence $ZC := \bigcup_{z\in Z} zC = C$.

It follows that if $C \supseteq \bigcap_{i\in\omega} D_i$, where each D_i is dense and open, then $C \supseteq \bigcap_{i\in\omega} ZD_i$, and each ZD_i is also dense and open. By 3.2.6, the sets $\widehat{ZD_i}$ are dense and open.

We argue that $\widehat{\bigcap_{i\in\omega} ZD_i} = \bigcap_{i\in\omega} \widehat{ZD_i}$. For $\subseteq$, if $x \in \widehat{\bigcap_{i\in\omega} ZD_i}$, then there is $h \in \bigcap_{i\in\omega} ZD_i$ with $x = \hat{h}$. But now $h \in ZD_i$ for all i, so $x \in \widehat{ZD_i}$ for all i, so $x \in \bigcap_{i\in\omega} \widehat{ZD_i}$.

For the reverse inclusion, suppose $x \in \bigcap_{i\in\omega} \widehat{ZD_i}$, with $x = \hat{h}$. Then for all i, $x \in \widehat{ZD_i}$, so for all i, $h \in ZD_i$. Hence $h \in \bigcap_{i\in\omega} ZD_i$, so $x \in \widehat{\bigcap_{i\in\omega} ZD_i}$.

So $\hat{C} \supseteq \widehat{\bigcap_{i\in\omega} ZD_i} = \bigcap_{i\in\omega} \widehat{ZD_i}$. Hence $\hat{C}$ contains a countable intersection of dense open sets, so it is comeagre, i.e. $\hat{g}$ is a generic. $\square$

Therefore $\mathrm{PSp}(V)$, $\mathrm{PU}(V)$ and $\mathrm{PO}(V)$ all contain a generic automorphism. By 1.5.2, this means that they are $\exists$ definable in $\mathrm{P\Gamma Sp}(V)$, $\mathrm{P\Gamma U}(V)$ and $\mathrm{P\Gamma O}(V)$ respectively. They are also normal in these groups, and the weak $\forall\exists$ interpretations that we shall find will satisfy the hypothesis of 1.5.1. Hence a weak $\forall\exists$ interpretation for each of $\mathrm{PSp}(V)$, $\mathrm{PU}(V)$ and $\mathrm{PO}(V)$ will suffice for reconstructing all the structures on $\mathrm{PG}(V)$ induced by groups respecting forms.

Remark 3.3.10 Propositions 3.3.8 and 3.3.9 are very close to results implicit in [9], and may well follow from that paper. However, the existence of ample generic automorphisms in the sense of [9] does not formally imply the existence of generics in our sense (generics in [9] may be over parameters).

3.3.2 The interpretation for $\mathrm{PSp}(V)$

The following facts will yield a weak $\forall\exists$ interpretation for $\mathrm{PSp}(V)$ acting on $\mathrm{PG}(V)$:

Proposition 3.3.11 $\mathrm{Sp}(V)$ *is transitive on the points of* $\mathrm{PG}(V)$.

Proof This is a consequence of Witt's theorem. $\square$

The following is well known (cf. [27], p. 71), and holds both in the finite- and $\aleph_0$-dimensional cases:

Lemma 3.3.12 *Let* $\tau \in \mathrm{GL}(V)$ *be a transvection. Then* $\tau \in \mathrm{Sp}(V)$ *if and only if* $U_\tau = d^\perp$.

Proof Let $\tau = \tau_{d,u}$ where $d \in V \setminus \{0\}$. Then:

$$\begin{aligned} \tau_{d,u} \in \mathrm{Sp}(V) \quad &\iff \quad \forall v, w \in V \ \beta(v,w) = \beta(v\tau_{d,u}, w\tau_{d,u}) \\ &= \quad \beta(v + u(v)d, w + u(w)d) \end{aligned}$$

$$= \quad \beta(v,w)+u(v)\beta(d,w)+u(w)\beta(v,d).$$

Therefore we need $u(v)\beta(d,w)+u(w)\beta(v,d)=0$ for all $v,w\in V$. We can choose $v\in V$ with $\beta(d,v)=1$. Then for all w we have $u(w)=u(v)\beta(d,w)$, that is $\ker(u)=d^{\perp}$. $\square$

Proposition 3.3.13 *There is a conjugacy class* $T=\tau_{d,u}^{\mathrm{Sp}(V)}$ *in* $\mathrm{Sp}(V)$ *such that for all* $\langle v\rangle\in\mathrm{PG}(V)$*, there is* $\tau_{d',u'}\in T$ *with* $\langle d'\rangle=\langle v\rangle$*.*

Proof First note that the conjugate of a symplectic transvection is a symplectic transvection: let $\tau_{d,u}\in\mathrm{Sp}(V)$ be a transvection, and let $g\in\mathrm{Sp}(V)$. Then, by 3.2.3, $\tau_{d,u}^{g}=\tau_{d^g,g^{-1}u}$. Since $\ker(u)=d^{\perp}$, we have that $(\ker(u))^g=(d^{\perp})^g$. But $(\ker(u))^g=\ker(ug)$ and $(d^{\perp})^g=(dg)^{\perp}$, so $\tau_{d^g,g^{-1}u}$ is a symplectic transvection as required. The claim then follows because $\mathrm{Sp}(V)$ is transitive on the points of $\mathrm{PG}(V)$. $\square$

Proposition 3.3.13 ensures that if we work with $\hat{C}=\hat{\tau}_{d,u}^{\mathrm{PSp}(V)}$, where $\hat{\tau}_{d,u}$ is a projective symplectic transvection, each point in $\mathrm{PG}(V)$ will be represented by at least one element of $\hat{C}$.

Lemma 3.3.12 enables us to find a simpler interpretation for $\mathrm{PSp}(V)$ than the one for $\mathrm{PGL}(V)$: since the direction of a transvection determines its fixed hyperplane uniquely, we can use Proposition 3.2.9 to identify those symplectic transvections that fix the same direction:

Proposition 3.3.14 *Let* $\hat{\tau},\hat{\sigma}$ *be projective symplectic transvections. Then:*

$$\hat{\tau}\hat{\sigma} \text{ is a projective symplectic transvection} \iff \langle d_\tau\rangle=\langle d_\sigma\rangle.$$

Proof This is a direct consequence of 3.2.9 and 3.3.12. $\square$

It follows that the relation "having the same direction" on the conjugacy class of projective symplectic transvections is indeed an $\exists$ definable equivalence relation in the language of groups.

3.3.3 A reconstruction result for $\mathrm{PU}(V)$ and $\mathrm{PO}(V)$

Our reconstruction results for the unitary and orthogonal spaces will involve selecting a suitable subset of V on which $\mathrm{U}(V)$ and $\mathrm{O}(V)$ are closed automorphism groups, and extending the interpretation to the full domain.

Fact 3.3.15 *The unitary space (V, β) has a basis of isotropic vectors. Moreover, $\mathrm{PU}(V)$ is transitive on the set of isotropic points of $\mathrm{PG}(V)$.*

Proof [27] pp. 116–117 and Theorem 10.12. □

Fact 3.3.16 *There is an orbit P of the orthogonal group $\mathrm{O}(V)$ on (V, Q) which consists of nonsingular vectors and contains a basis for V.*

Proof It is known that the orthogonal group $\mathrm{O}(V)$ is irreducible in its natural action on V, so any orbit spans V. In particular if $v \in V$ is nonsingular, then $\{v^g : g \in \mathrm{O}(V)\}$ consists of nonsingular vectors and it contains a basis, as required. □

We now prove that $\mathrm{PO}(V)$ acting on an orbit $\hat{P}$ of nonsingular points (resp. $\mathrm{PU}(V)$ acting on the set $\hat{P}$ of isotropic points) is closed, and that $\mathrm{PG}(V) = \mathrm{dcl}(\hat{P})$.

Lemma 3.3.17 *Let $\mathcal{M}$ be a first order structure, W a set, and $\pi : \mathcal{M} \to W$ be a finite-to-one surjection whose fibres form an $\mathrm{Aut}(\mathcal{M})$-invariant partition of $\mathcal{M}$. Let $\mu : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Sym}(W)$ be the map defined by $\mu(g) = ((w)\pi^{-1}g)\pi$ for all $g \in \mathrm{Aut}(C)$ and $w \in W$. Then μ maps closed subgroups of $\mathrm{Aut}(\mathcal{M})$ to closed subgroups of $\mathrm{Sym}(W)$.*

Proof [14], 1.4.2. □

Proposition 3.3.18 *Let $\mathcal{M}$ be a structure, $G = \mathrm{Aut}(\mathcal{M})$ and $P \subseteq \mathcal{M}$ be a G-invariant subset such that $\mathcal{M} = \mathrm{dcl}(P)$. Then G is closed on P.*

Proof Suppose that $g \in G$. Then, since $P^g = P$ and g is a bijection on $\mathcal{M}$, g is also a bijection on P.

Recall that G is closed in $\mathrm{Sym}(P)$ if and only if the following holds: if $g \in \mathrm{Sym}(P)$ is such that for all $\bar{p} \in P^n$ there is $h \in G$ such that $\bar{p}^h = \bar{p}^g$, then $g \in G$. So let $g \in \mathrm{Sym}(P)$ be as in the hypothesis, i.e. g behaves like an element of G on each finite tuple in P. We want to show that $g \in G$.

Extend g to $g' \in \mathrm{Sym}(\mathcal{M})$ as follows: for $m \in \mathcal{M}$, let $m \in \mathrm{dcl}(\bar{p})$, $\bar{p} \in P^k$, be defined by the formula $\phi(x, \bar{p})$. Choose $h \in G$ agreeing with g on $\bar{p}$, and extend g to g' defined by

$$m^{g'} := \phi(\mathcal{M}, \bar{p}^h).$$

Then g' is well-defined: if $m = \phi(\mathcal{M}, \bar{p})$ and $m = \psi(\mathcal{M}, \bar{q})$, then $\phi(\mathcal{M}, \bar{p}) = \psi(\mathcal{M}, \bar{q})$ implies that $\phi(\mathcal{M}, \bar{p}^h) = \psi(\mathcal{M}, \bar{q}^h)$. It is easy to see that g' is independent of the choice of h.

Now let $\bar{m} \in \mathcal{M}^n$, and let ψ be any n-formula. Suppose $\{m_i\} = \phi_i(\mathcal{M}, \bar{p}^i)$ for $i = 1, \ldots, n$. For each $i = 1, \ldots, n$ there is a 0-definable partial function f_i such that $m_i = f_i(\bar{p}^i)$. Then

$$\begin{aligned} \mathcal{M} \models \psi(\bar{m}) &\iff \mathcal{M} \models \psi(f_1(\bar{p}^1), \ldots, f_n(\bar{p}^n)) \\ &\iff \mathcal{M} \models \psi(f_1((\bar{p}^1)^h), \ldots, f_n((\bar{p}^n)^h)) \\ &\iff \mathcal{M} \models \psi(\bar{m}^{g'}). \end{aligned}$$

Hence $g' \in \mathrm{Aut}(\mathcal{M})$, as required. $\square$

Proposition 3.3.19 *Let* $(\mathrm{PG}(V), \beta, Q)$ *be the projective unitary (resp. orthogonal) space, and* $\hat{P}$ *be the set of isotropic (resp. an orbit of nonsingular) 1-dimensional subspaces. Let* $\mathcal{O}$ *be an orbit of* $G = \mathrm{PO}(V, \beta, Q)$ *on* $(\mathrm{PG}(V), \beta, Q)$. *Then* $\mathcal{O} \subseteq \mathrm{dcl}(\hat{P})$. *It follows that* G *is faithful on* $\hat{P}$.

Proof Let $\mathcal{O}$ be as in the statement. We know that the pre-image P of $\hat{P}$ under $\widehat{\ }$ contains a basis for V, so every $v \in V$ is a linear combination of vectors in P. Let $\mathcal{O} = \langle v \rangle^{O(V,\beta,Q)}$, and suppose that $\langle v \rangle = \langle \alpha_1 v_1 + \ldots + \alpha_r v_r \rangle$, $\alpha_i \in F$, $v_i \in P$.

If $f \in G$ fixes $\langle v_1\rangle, \dots, \langle v_r\rangle$, then $v_1, \dots v_r$ have finitely many translates in V, hence $v_1 + \cdots + v_r$ has finitely many translates. So $\langle v\rangle \in \mathrm{acl}(\langle v_1\rangle, \dots, \langle v_r\rangle)$. So we have that $\mathcal{O} \subseteq \mathrm{acl}(\hat{P})$.

Suppose for a contradiction that there is $v \in \mathcal{O}$ such that $v \notin \mathrm{dcl}(\hat{P})$. Then, by a König's Lemma argument, there is $g \in G_{\hat{P}}$ such that $vg \neq v$. But then $G_{\hat{P}}$ is normal in G, closed and nontrivial, since it contains g. But, since Theorem 1 in [15] implies that G has no proper non trivial closed normal subgroups, this is a contradiction.

It follows that if $g \in G_{\hat{P}}$, then $\langle v\rangle g = v$ for all $\langle v\rangle \in \mathrm{PG}(V)$, so $g = \mathrm{id}$, i.e. G is faithful. □

Corollary 3.3.20 *Let P be the set of isotropic vectors in the unitary space (V, β), resp. an orbit of nonsingular vectors in the orthogonal space (V, O). Then $G =$ $\mathrm{O}(V)$ (resp. $G = \mathrm{U}(V)$) is closed in its action on P. It follows that the projective image $\hat{G}$ of G is closed in its action on $\hat{P} := \{\langle v\rangle : v \in P\}$.*

Proof By 3.3.18 and 3.3.19, G is closed on P. By 3.3.17 with $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}\rangle =$ $\langle G, P\rangle$ and $W = \hat{P}$, the projective image $\hat{G}$ of G is closed in its action on $\hat{P}$. □

Hence $\mathrm{PO}(V)$ and $\mathrm{PU}(V)$ induce the automorphism group of a structure on an orbit of nonsingular 1-subspaces and on the set of isotropic 1-subspaces respectively. We shall start by looking for weak $\forall\exists$ interpretations for the structures $\langle \mathrm{PO}(V, \beta, Q), \hat{P}\rangle$ and later extend our results to $\langle \mathrm{PO}(V, \beta, Q), \mathrm{PG}(V)\rangle$.

Fact 3.3.21 *Suppose $\tau_{d,u}$ is a transvection in $\mathrm{GL}(V)$. Then $\tau_{d,u} \in \mathrm{U}(V)$ if and only if it is of the form*

$$\tau(v) = v + a\beta(v, d)d$$

where d is isotropic and $a \in F$ satisfies $a + a^\sigma = 0$. In particular, for each isotropic vector d there is a unitary transvection having direction $\langle d\rangle$.

Proof [27], pp. 118–119. □

Projective unitary transvections are defined in the usual way. Note that here, as in the symplectic case, for a transvection $\tau_{d,u}$ we have $ker(u) = \langle d \rangle^{\perp} = d^{\perp}$, so our weak $\forall\exists$ interpretation for $\langle \mathrm{PU}(V), \hat{P} \rangle$ is based on the same formula as we used in the symplectic case.

Proposition 3.3.22 *There is a conjugacy class $T = \hat{\tau}_{d,u}^{\mathrm{PU}(V)}$ in* $\mathrm{PU}(V)$ *such that for all isotropic $\langle v \rangle \in \mathrm{PG}(V)$, there is $\tau_{d',u'} \in T$ with $\langle d' \rangle = \langle v \rangle$.*

Proof The proof is similar to 3.3.13. $\square$

Proposition 3.3.23 *Let $\hat{\tau}$, $\hat{\sigma}$ be projective unitary transvections. Then*

$$\hat{\sigma}\hat{\tau} \textit{ is a unitary projective transvection} \iff \langle d_\tau \rangle = \langle d_\sigma \rangle.$$

Proof This is a consequence of the fact that for $\tau_{d,u}$ $\ker(u) = \langle d \rangle^{\perp} = d^{\perp}$ and of 3.2.9. $\square$

The reconstruction result for the orthogonal space is very similar to the unitary case, except that when $\mathrm{char}(F) \neq 2$ there are no transvections in $\mathrm{O}(V)$ so we use reflections instead, and we need a basis of nonsingular, rather than isotropic, vectors. Let us deal with the characteristic 2 case first:

Lemma 3.3.24 *If* $\mathrm{char}(F) = 2$*, the following hold:*

1. *the orthogonal space (V, Q) contains a transvection τ;*
2. *$v\tau = v + Q(v)^{-1}u$ for a nonsingular vector u;*
3. *each nonsingular point in* $\mathrm{PG}(V)$ *is the centre of a unique transvection.*

Proof [2], 22.3. $\square$

So the even characteristic case is treated like the unitary case, except that, by virtue of 3.3.24 2. above, there is no need to quotient the conjugacy class of orthogonal transvections by an equivalence relation. For the general case, we need to define *reflections*.

Definition 3.3.25 *A* ***reflection*** *in* $\mathrm{O}(V, Q)$ *is a map of the form*

$$\tau_u(v) = v - Q(u)^{-1}\beta(v, u)u$$

where u *is a nonsingular vector. We call* $\langle u \rangle$ *the* ***centre*** *of* τ_u.

Note that τ_u fixes $\langle u \rangle^{\perp}$. Moreover, for every nonsingular vector $u \in (V, Q)$ there is a unique reflection with centre $\langle u \rangle$:

$$\begin{aligned} v\tau_{\lambda u} &= v - Q(\lambda u)^{-1}\beta(v, \lambda u)\lambda u \\ &= v - \frac{Q(\lambda u)^{-1}}{\lambda^2}\lambda^2\beta(v, u)u \\ &= v\tau_u. \end{aligned}$$

Definition 3.3.26 *A* ***projective reflection*** *is an element of* $\mathrm{PO}(V, Q)$ *of the form* $\hat{\tau}_u$ *where* τ_u *is a reflection.*

It follows easily from the above that for every nonsingular point of $\mathrm{PG}(V)$ there is a unique projective reflection with centre $\langle u \rangle$.

Proposition 3.3.27 *For each orbit* P *of* $\mathrm{O}(V)$ *consisting of nonsingular vectors there is a conjugacy class* $C \subseteq \mathrm{O}(V)$ *consisting of reflections such that for all* $v \in P$ *there is a unique reflection in* C *having centre* $\langle v \rangle$. *It follows that there is a bijection between the conjugacy class* $\hat{C} \subseteq \mathrm{PO}(V)$ *and the orbit* $\hat{P}$ *such that* $\langle \mathrm{PO}(V), \hat{P} \rangle \cong \langle \mathrm{PO}(V), \hat{C} \rangle$.

Proof Let $\tau_u \in \mathrm{O}(V, Q)$ be a reflection. Then

$$\begin{aligned} (v)g^{-1}\tau_u g &= (vg^{-1} - Q(u)^{-1}\beta(vg^{-1}, u)u)g \\ &= v - Q(u)^{-1}\beta(vg^{-1}, u)ug) \\ &= v - Q(ug)^{-1}\beta(v, ug)ug) \\ &= v\tau_{ug}. \end{aligned}$$

So the conjugate by $g \in \mathrm{O}(V)$ of a reflection with centre u is a reflection of centre u^g. Since $\mathrm{O}(V)$ is transitive on the orbit P, and by the remark following 3.3.26, the claim follows. $\square$

The facts above yield a weak $\forall\exists$ interpretation for $\mathrm{PO}(V)$ acting on an orbit $\hat{P}$ of nonsingular points of $\mathrm{PG}(V)$. It is clear that in this case we do not need to find an equivalence relation on the conjugacy class considered, since there is naturally a bijection with the orbit $\hat{P}$.

So far we have obtained weak $\forall\exists$ interpretations for $\langle \mathrm{PU}(V), \hat{P}\rangle$, where $\hat{P}$ is the set of isotropic points in the projective unitary space $(\mathrm{PG}(V), \beta)$, and for $\langle \mathrm{PO}(V), \hat{P}\rangle$, where $\hat{P}$ is an orbit of nonsingular points in the orthogonal projective space $(\mathrm{PG}(V), Q)$. By 3.3.19, this gives a generalised weak $\forall\exists$ interpretation for $\langle \mathrm{PO}(V), \mathrm{PG}(V)\rangle$ and $\langle \mathrm{PU}(V), \mathrm{PG}(V)\rangle$.

Proposition 3.3.19 gives a weak $\forall\exists$ interpretation in the sense of 3.0.13 for $\mathrm{PO}(V)$ and $\mathrm{PU}(V)$ acting on $\mathrm{PG}(V)$. In order to lift these interpretations to $\mathrm{P\Gamma U}(V)$ and $\mathrm{P\Gamma O}(V)$ and to the intermediate closed subgroups, we prove the following extension of Proposition 3.3.19.

Proposition 3.3.28 *Let G such that $\mathrm{PU}(V) \leq G \leq \mathrm{P\Gamma U}(V)$ (resp. $\mathrm{PO}(V) \leq G \leq \mathrm{P\Gamma O}(V)$) be a closed group on the set $\hat{P}$ of isotropic (resp. on an orbit of nonsingular) 1-dimensional subspaces of V. Let $\mathcal{O}$ be an orbit of G on $\mathrm{PG}(V)$. Then $\mathcal{O} \subseteq \mathrm{dcl}(\hat{P})$. It follows that G is faithful on $\hat{P}$.*

Proof For ease of notation, we shall state the argument for $\mathrm{PU}(V) \leq G \leq \mathrm{P\Gamma U}(V)$. The case $\mathrm{PO}(V) \leq G \leq \mathrm{P\Gamma O}(V)$ is entirely similar. We know that $\mathrm{PU}(V) \lhd \mathrm{P\Gamma U}(V)$, and that $|\mathrm{P\Gamma U}(V) : \mathrm{PU}(V)|$ is finite, therefore $|G : \mathrm{PU}(V)|$ is also finite. Also, G is transitive on $\hat{P}$.

We claim that for G acting on $\mathrm{PG}(V)$, $\mathcal{O} \subseteq \mathrm{acl}(\hat{P})$. By 3.3.19, we know that for all $p \in \mathcal{O}$ there is $\bar{q} \in \bar{P}$ such that $\mathrm{PU}(V)_{\bar{q}}$ fixes p. We want to prove that p has finitely many translates under $G_{\bar{q}}$. This is equivalent to proving that $|G_{\bar{q}} : G_{\bar{q}p}| < \aleph_0$. Suppose for a contradiction that there are $(g_i : i \in \omega)$ which all lie in different cosets of $G_{\bar{q}p}$ in $G_{\bar{q}}$. Then the elements $g_i g_j^{-1}$ are all in $G_{\bar{q}}$ but not in $G_{\bar{q}p}$, hence they are not in $\mathrm{PU}(V)$. So we get that the $g_i, i \in \omega$ all lie in different cosets of $\mathrm{PU}(V)$ in G, which contradicts the fact that $|G : \mathrm{PU}(V)|$ is finite.

Next we show that $\mathcal{O} \subseteq \mathrm{dcl}(\hat{P})$. Suppose for a contradiction that $\mathcal{O}$ is not definable over $\hat{P}$. Then there is $g \in G$, $g \neq \mathrm{id}$ such that $g|_{\hat{P}} = \mathrm{id}$ (as in 3.3.19, by a König's lemma argument). It follows that $G_{\hat{P}}$ is nontrivial. Since $\hat{P}$ is an orbit, $G_{\hat{P}} \lhd G$. But $G_{\hat{P}} \leq G_p$ for any $p \in \hat{P}$. Since $|G : G_p| = |\cos(G : G_p)| = |\hat{P}| = \aleph_0$, $|G : G_{\hat{P}}|$ is infinite. But this is a contradiction, as G has no closed normal subgroups of infinite index. Indeed, if $H \lhd G$ is a closed nontrivial normal subgroup of infinite index, then $H \cap \mathrm{PU}(V)$ is a proper nontrivial closed normal subgroup of $\mathrm{PU}(V)$, a contradiction by [15]. Faithfulness of G follows as in 3.3.19. $\square$

Corollary 3.3.29 *If G is a closed group acting on $\mathrm{PG}(V)$ such that $\mathrm{PU}(V) \leq G \leq \mathrm{P\Gamma U}(V)$ (resp. $\mathrm{PO}(V) \leq G \leq \mathrm{P\Gamma O}(V)$), then $\langle G, \mathrm{PG}(V)\rangle$ has a generalised weak $\forall\exists$ interpretation.*

Proof By 3.3.20, 3.3.22, 3.3.23, 3.3.27, 3.3.19, there is a weak $\forall\exists$ interpretation for $\langle \mathrm{PU}(V), \hat{P}\rangle$ (resp. $\langle \mathrm{PO}(V), \hat{P}\rangle$). Since $\mathrm{PU}(V) \lhd \mathrm{P\Gamma U}(V)$ (resp. $\mathrm{PO}(V) \lhd \mathrm{P\Gamma O}(V)$), we can apply 1.5.1 to obtain a weak $\forall\exists$ interpretation for $\langle G, \hat{P}\rangle$. By 3.3.28, this yields a generalised weak $\forall\exists$ interpretation for $\langle G, \mathrm{PG}(V)\rangle$. $\square$

3.4 A reconstruction result for affine spaces

In what follows we shall give an interpretability result for the general case of a primitive ω-categorical structure whose automorphism group has a nontrivial abelian normal subgroup. This result applies to the affine group $\mathrm{AGL}(V)$ of affine transformations of V, and it proves that V as an affine space is interpretable in $\mathrm{AGL}(V)$. We assume V to be ω-dimensional over a finite field F, as before.

Let us recall the basic definitions and notation about the affine group $\mathrm{AGL}(V)$. An affine transformation on V is a map $T_{M,b}$ of the form

$$vT_{M,b} := vM + b$$

where $M \in \mathrm{GL}(V)$ and $b \in V$. Then $\mathrm{AGL}(V)$ is the group of affine transformations on V. The affine group acts on V in the obvious way. Moreover, $\langle \mathrm{AGL}(V), V\rangle$ is an ω-categorical structure and the action of $\mathrm{AGL}(V)$ on V is primitive and faithful.

The affine transformations of the form $T_{I,b}$ where I is the identity in $\mathrm{GL}(V)$ are called the **translations** and they form a normal subgroup $\mathrm{T}(V) \lhd \mathrm{AGL}(V)$. Also, the multiplicative group $\mathrm{T}(V)$ is isomorphic to the additive group V, so $\mathrm{T}(V)$ is abelian. By identifying $T_{M,0} \in \mathrm{AGL}(V)$ with $M \in \mathrm{GL}(V)$ and $T_{I,b} \in \mathrm{T}(V)$ with $b \in V$ it is easy to see that every element of $\mathrm{AGL}(V)$ can be expressed uniquely as the product of an element of $\mathrm{GL}(V)$ and an element of V. Moreover, $\mathrm{GL}(V) = \mathrm{Stab}_{\mathrm{AGL}(V)}(0)$, so $\mathrm{GL}(V) \leq \mathrm{AGL}(V)$, so we can write

$$\begin{aligned} \mathrm{AGL}(V) &= \mathrm{T}(V) \rtimes \mathrm{GL}(V) \\ &= V \rtimes \mathrm{GL}(V). \end{aligned}$$

This will be proved in Proposition 3.4.4 below.

We shall give our interpretability result in the general setting of an oligomorphic primitive permutation group G acting on a countable set X and having an abelian normal subgroup A. We shall show that then the structure on X is interpretable in G. This result applies to the affine group if we take $G = \mathrm{AGL}(V)$, $A = \mathrm{T}(V)$ and $X = V$. We start with some folklore proofs, some of which can be found in [4].

Lemma 3.4.1 *If X is a transitive G-space and $A \lhd G$, then the orbits of A are the equivalence classes of a G-invariant equivalence relation on X.*

Proof Define $\sim$ on X by: $\alpha \sim \beta :\Longleftrightarrow \alpha^a = \beta$ for some $a \in A$. It is easy to see that $\sim$ is an equivalence relation. We claim further that $\sim$ is G-invariant. Let $\alpha \sim \beta$ and let $g \in G$. We claim that $\alpha^g \sim \beta^g$.

$$\begin{aligned} \alpha \sim \beta &\Longleftrightarrow \alpha^a = \beta \text{ for some } a \in A \\ &\Longleftrightarrow (\alpha^a)^g = \beta^g \\ &\Longleftrightarrow \alpha^{ag} = \beta^g \end{aligned}$$

By normality of A, there is some $b \in A$ such that $ag = gb$. Hence $\alpha^{gb} = \beta^g$, which implies that $\alpha^g \sim \beta^g$ as required. $\square$

Lemma 3.4.2 *If G is faithful and primitive on X and $A \lhd G$ is non trivial, then A is transitive.*

Proof By 3.4.1, the orbits of A on X form a G-congruence. As G is primitive, this congruence is either trivial or improper. If it is trivial, then A fixes every element of X. But G is faithful and A is nontrivial, so the congruence is improper, that is, A is transitive. $\square$

Lemma 3.4.3 *Any transitive abelian permutation group H acting faithfully on a set X is regular.*

Proof Let $h \in H$ be such that $x^h = x$ for $x \in X$. Then for any non identity $g \in H$ we have $x^{gh} = x^{hg} = x^g$. By transitivity, h fixes all $x \in X$. By faithfulness, $h = \mathrm{id}$. $\square$

So if $A \lhd G$ is a non trivial normal subgroup, then A is transitive on X (by 3.4.2). If A is also abelian, then by 3.4.3 A is regular on X.

Proposition 3.4.4 *Let G be a primitive faithful group acting on a set X, and let A be a non trivial abelian normal subgroup. Let $\alpha \in X$ and let G_α be the stabiliser of α. Then $G = A \rtimes G_\alpha$.*

Proof For $G = A \rtimes G_\alpha$ we need to show

1. $AG_\alpha = G$;
2. $A \cap G_\alpha = \{1\}$.

For 1., let $g \notin G_\alpha$. Then $G_\alpha g \neq G_\alpha$. Pick $a \in A$ such that $\alpha^a = \alpha^g$ (a exists because A is transitive on X). Then $G_\alpha a = G_\alpha g$, hence $a^{-1}g \in G_\alpha$. So $g = aa^{-1}g$, with $a \in A$ and $a^{-1}g \in G_\alpha$, as required.

For 2., let $g \in A \cap G_\alpha$. Then $\alpha^g = \alpha$. Take any $a \in A$. Then $ag = ga$ so $\alpha^{ga} = \alpha^a = \alpha^{ag}$, therefore $g \in G_{\alpha^a}$. Since A is transitive, as a ranges over A, α^a ranges over the whole of X. Therefore $g \in G_x$ for all x in X, i.e. g fixes every element of X. By regularity of A, $g = 1$. $\square$

We now show that a suitable identification allows us to regard A as a copy of X in the group G.

Proposition 3.4.5 *Let X, G, A and α be as above. Then*

$$(G_\alpha, X) \cong (G_\alpha, A),$$

where G_α acts on A by conjugation.

Proof Consider the map $\theta : A \to X$ defined by

$$\theta : 1 \to \alpha$$

$$\theta : g \to \alpha^g.$$

We claim θ defines an isomorphism between the natural action of G_α on X and the action of G_α on A by conjugation.

First note that by 3.4.4 2. we have:

$$\begin{aligned} \alpha^g = \alpha^h \ &\Rightarrow \ \alpha^{gh^{-1}} = \alpha \\ &\Rightarrow \ gh^{-1} \in G_\alpha \cap A \\ &\Rightarrow \ gh^{-1} = 1 \\ &\Rightarrow \ g = h \end{aligned}$$

so θ is injective. Since A is transitive, θ is also surjective.

Since $\theta(a^g) = \alpha^{g^{-1}ag} = \alpha^{ag}$ (as $g^{-1} \in G_\alpha$) $= [\theta(a)]^g$, θ is also a G_α-morphism, as required. $\square$

Proposition 3.4.6 *Let G, X, A and α be as above, and suppose further that G is oligomorphic on X. Then X with its structure is interpretable with parameters in G.*

Proof We start by showing that the set X is definable in G. Note that since G is primitive, X has no non trivial proper blocks, i.e. no non trivial proper subset Y such that for all $g \in G$ either $Y^g = Y$ or $Y \cap Y^g = \emptyset$. Via the identification of X and A given in the proof of 3.4.5, this means that A has no non trivial proper subgroups that are G-invariant. So A is minimal among non trivial normal subgroups.

So choose $g \in A$, $g \neq 1$. We claim that $A = \{\prod_{i\in I}(g^{\varepsilon_i})^{h_i} : h_i \in G,\ \varepsilon_i = \pm 1\}$.

Let $H = \{\prod (g^{\varepsilon_i})^{h_i}\}$: clearly, $\{1\} \neq H \leq G$ and $H \subseteq A$. Now pick $h \in G$ and $\prod (g^{\varepsilon_i})^{h_i} \in H$. Then $(\prod (g^{\varepsilon_i})^{h_i})^h = \prod (g^{\varepsilon_i})^{h_i h} \in H$. So $H \lhd G$. By minimality of A among non trivial normal subgroups, $H = A$.

We now claim that there is a bound on the number of conjugates of g into which an element of A factors. We know that G is oligomorphic on X hence it is oligomorphic in its action on A as a pure set inherited from X via the identification of X and A, so in particular it has a finite number of orbits on A^2. Therefore the centraliser $C_G(g)$ has finitely many orbits on A. Now, elements which require a different number of products of conjugates of g lie in different orbits of $C_G(g)$. Our claim follows, and A is definable.

Now consider the map $\phi : G \to G_\alpha$ given by $\phi(g) = \phi(ah) = h$, where $a \in A, h \in G_\alpha$ are the unique decomposition of g as an element of the semidirect product $A \rtimes G_\alpha$. It is easy to check that ϕ is an epimorphism with kernel A, so that

$$G/A \cong G_\alpha.$$

We define an action of $A \rtimes G/A$ on A as follows:

$$a^{bAh} := (ab)^h \text{ for all } a, b \in A, Ah \in G/A.$$

For ease of notation, we shall write bh (rather than bAh) for the general element of $A \rtimes G/A$. Then:

$$(a^{bh})^{ck} = (h^{-1}abh)^{ck} = k^{-1}h^{-1}abhck$$

and

$$a^{bhck} = a^{bhch^{-1}hk} = k^{-1}h^{-1}abhch^{-1}hk = k^{-1}h^{-1}abhck$$

so that we have indeed defined an action. By using the isomorphism between G_α and $A \rtimes G/A$, we can identify the actions $\langle A \rtimes G/A, A\rangle$ and $\langle A \rtimes G_\alpha, A\rangle$. Then the structure on X is given by the orbits of $A \rtimes G/A$ on A^n for all $n \in \omega$. $\square$

If G_α were definable in G, it would be easy to get an interpretation for $\langle G, X\rangle$, via the isomorphism $\langle G, X\rangle \cong \langle G, \cos(G : G_\alpha)\rangle$. But it is not obvious that G_α is definable, and that is why we turn to the action of $A \rtimes G/A$ instead.

Proposition 3.4.6 applies to many subgroups of G. Indeed, it applies to all the primitive smoothly approximable structures of affine type described in [19].

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