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RECONSTRUCTION OF CLASSICAL GEOMETRIES FROM THEIR AUTOMORPHISM GROUP

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Abstract

Let V be a countably infinite dimensional vector space over a finite field F. Then V is ω-categorical, and so are the projective space $\mathrm{PG}(V)$ and the projective symplectic, unitary and orthogonal spaces on V. Using a reconstruction method developed by M. Rubin we prove the following result: let $\mathcal{M}$ be one of the spaces above, and let $\mathcal{N}$ be an ω-categorical structure such that $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$ as abstract groups. Then $\mathcal{M}$ and $\mathcal{N}$ are bi-intepretable. We also give a reconstruction result for the affine group $\mathrm{AGL}(V)$ acting on V by proving that V as an affine space is interpretable in $\mathrm{AGL}(V)$.

1. *Introduction*

Reconstruction results are intended to answer a very natural question: if we know the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a first order structure $\mathcal{M}$, how much do we know about $\mathcal{M}$? The answer depends both on the extent to which we know $\mathrm{Aut}(\mathcal{M})$, and the extent to which we want to recover $\mathcal{M}$: we may want $\mathrm{Aut}(\mathcal{M})$ to determine $\mathcal{M}$ up to bi-interpretability, or up to isomorphism.

A very natural class for investigating this question is the class of ω-categorical structures: the Ryll-Nardzewski theorem ensures that the automorphism groups of these structures are very rich. Here the standard topology on $\mathrm{Aut}(\mathcal{M})$ — that generated by the stabilizers of finite tuples of $\mathcal{M}$ and their cosets — becomes important: if two ω-categorical structures have automorphism groups which are isomorphic as topological groups, then they are bi-interpretable. If the automorphism groups are isomorphic as permutation groups, then the two structures are bi-definable. Reconstruction techniques for ω-categorical structures generally seek conditions on $\mathrm{Aut}(\mathcal{M})$ so that the pure group determines the topology (this is the case if a structure has the small index property) or its action on $\mathcal{M}$.

The reconstruction method used in this paper has been developed by Mati Rubin in [**13**], and falls into the latter category. We shall give an application of Rubin's method of weak $\forall\exists$ interpretations to obtain reconstruction results for the projective space $\mathrm{PG}(V)$, where V is a vector space of dimension ω over a finite field F, and for the projective symplectic, unitary and orthogonal spaces on V. The last section of the paper contains a reconstruction result for various subgroups of the affine group $\mathrm{AGL}(V)$ acting on V: we show that V, as an affine space, is definable in $\mathrm{AGL}(V)$.

Rubin's main result gives a reconstruction criterion for the class of ω-categorical structures without algebraicity. Whenever such a structure $\mathcal{M}$ has a so called *weak $\forall\exists$ interpretation* and $\mathcal{N}$ is ω-categorical without algebraicity, it is enough to know $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$ in order to conclude that the permutation groups $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}\rangle$ and $\langle \mathrm{Aut}(\mathcal{N}), \mathcal{N}\rangle$ are isomorphic. Given an ω-categorical transitive structure $\mathcal{M}$,

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one selects a conjugacy class C in $\mathrm{Aut}(\mathcal{M})$ and a conjugacy invariant equivalence relation E on C that is $\forall\exists$ definable in the language of groups, so that the permutation groups $\langle \mathrm{Aut}(\mathcal{M}), C/E\rangle$ and $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}\rangle$ are isomorphic. If such C and E can be found, $\mathcal{M}$ is said to have a *weak* $\forall\exists$ *interpretation.* Generally (but not necessarily) C consists of automorphisms having a single fixed point and E is "having the same fixed point". So an element of $\mathcal{M}$ is identified with the equivalence class of automorphisms that fix it. When $\mathcal{M}$ is not transitive, a weak $\forall\exists$ interpretation for $\mathcal{M}$ consists of a weak $\forall\exists$ interpretation for each orbit of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{M}$. Let us state the formal definitions (to be found in [**13**]).

Let G be a group acting transitively on the countable set M, and let E be a G-invariant equivalence relation on M, i.e. such that

$$\forall\ a, b \in M, \forall\ g \in G\ aEb \Rightarrow a^g E b^g.$$

Then G has a natural action on the set of equivalence classes M/E, that is, $(a/E)^g = (a^g)/E$, where a/E is the equivalence class of $a \in M$.

Let now $\vec{g} = \langle g_1, \ldots, g_n\rangle \in G^n$, and let $\phi(\vec{g}, x, y)$ be a formula in the language of groups with parameters $\vec{g}$. Let $C = g_1^G$ be the conjugacy class of $g_1 \in G$. Then ϕ is an $\forall\exists$**-equivalence formula** for G if:

– ϕ is $\forall\exists$;
– Group theory $\vdash \forall\bar{u}(\phi(\bar{u}, x, y)$ is an equivalence relation on the conjugacy class of $u_1)$;
– for given $\bar{g}$, $\phi(\bar{g}, x, y)$ defines a conjugacy invariant equivalence relation on C.

We shall write E^ϕ for the equivalence relation defined by ϕ[†].

Definition 1.1 Weak $\forall\exists$ interpretation, transitive case. Let $\mathcal{M}$ be an ω-categorical structure such that $\mathrm{Aut}(\mathcal{M})$ acts transitively on $\mathcal{M}$. A **weak** $\forall\exists$ **interpretation** for $\mathcal{M}$ is a triple $\langle \phi, \vec{g}, \tau\rangle$, where ϕ is an $\forall\exists$-equivalence formula, $\vec{g} \in \mathrm{Aut}(\mathcal{M})^n$, τ is an isomorphism between the permutation groups $\langle \mathrm{Aut}(\mathcal{M}), C/E^\phi\rangle$ and $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}\rangle$, that is, $\tau : C/E^\phi \to \mathcal{M}$ is a bijection such that for all $g, h \in \mathrm{Aut}(\mathcal{M})$

$$[\tau(h/E^\phi)]^g = \tau(h^g/E^\phi).$$

By the Ryll-Nardzewski theorem, if $\mathcal{M}$ is ω-categorical then $\mathcal{M}^n$ is partitioned into a finite number of orbits of $\mathrm{Aut}(\mathcal{M})$, for every $n \in \mathbf{N}$. In particular, $\mathcal{M}$ is partitioned into finitely many orbits, corresponding to 1-types, and $\mathrm{Aut}(\mathcal{M})$ acts transitively on each of them. We can thus extend the definition of a weak $\forall\exists$ interpretation to the general case when $\mathcal{M}$ is not transitive.

Definition 1.2 Weak $\forall\exists$ interpretation. Let $\mathcal{M}$ be an ω categorical structure with 1-types $P_1, \ldots, P_n$. A **weak** $\forall\exists$ **interpretation** for $\mathcal{M}$ is a tuple $\langle \vec{\phi}, \vec{g}, \vec{\tau}\rangle$, where $\vec{\phi} = (\phi_1, \ldots, \phi_n)$ are $\forall\exists$ equivalence formulae, $\vec{g} = (\vec{g}^1, \ldots, \vec{g}^n)$ are tuples of elements of $\mathrm{Aut}(\mathcal{M})$, $\vec{\tau} = (\tau_1, \ldots, \tau_n)$ are maps such that each triple $\langle \phi_i, \vec{g}^i, \tau_i\rangle$ is a weak $\forall\exists$ interpretation for the structure induced on P_i.

[†]Rubin's original definition of an $\forall\exists$ equivalence formula ϕ requires ϕ to define an equivalence relation that is conjugacy invariant in **all** groups. However, it is not difficult to see that Rubin's theorem below works under the weaker assumption that the equivalence relation defined by ϕ is conjugacy invariant in the given group.

We can now state Rubin's main result.

THEOREM Rubin, 1987. *Let K be the class of ω-categorical structures without algebraicity. Let $\mathcal{M} \in K$ have a weak $\forall\exists$-interpretation, and let $\mathcal{N} \in K$ be such that* $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$ *as pure groups. Then* $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M}\rangle \cong \langle \mathrm{Aut}(\mathcal{N}), \mathcal{N}\rangle$*, that is, $\mathcal{M}$ and $\mathcal{N}$ are bi-definable.*

Dropping the assumption of absence of algebraicity, a weak $\forall\exists$-interpretation permits reconstruction up to bi-interpretability ([**13**], p. 227):

PROPOSITION 1.3. *Let $\mathcal{M}$, $\mathcal{N}$ be ω-categorical and let $\mathcal{M}$ have a weak $\forall\exists$ interpretation. Then: if* $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$*, $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable.*

REMARK 1.4. In this paper we work with a slight generalisation of Rubin's definition of weak $\forall\exists$ interpretation. First, we allow the conjugacy class to be a conjugacy class on tuples from the group, i.e. of the form $C = (g_1, \ldots, g_n)^G$. Second, we do not require that all the 1-types of $\mathcal{M}$ should have such a weak $\forall\exists$ interpretation; merely that there are 1-types $P_1, \ldots, P_r$ of $\mathcal{M}$ each of which has a weak $\forall\exists$ interpretation via a conjugacy class of tuples as above, such that

(i) $\mathcal{M} \subseteq \mathrm{dcl}(\{x : P_1(x) \vee \ldots \vee P_r(x)\})$, and

(ii) $\mathrm{Aut}(\mathcal{M})$ is faithful and closed in its action on $\{x : P_1(x) \vee \ldots \vee P_r(x)\}$.

From now on in this paper, "weak $\forall\exists$ interpretation" is used in this more general sense. Rubin's argument works for conjugacy classes of tuples, and it follows that if $\mathcal{M}$ and $\mathcal{N}$ are ω-categorical with isomorphic automorphism groups, and $\mathcal{M}$ has a weak $\forall\exists$ interpretation in this more general sense, then $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable.

We shall write $\mathrm{GL}(V)$ and $\mathrm{PG}(V)$ for $\mathrm{GL}(\omega, q)$ and $\mathrm{PG}(\omega, q)$ respectively, and similarly for symplectic, unitary and orthogonal groups and their projective versions.

The theorem we prove is the following:

THEOREM. *Let V be an ω-dimensional vector space over a finite field F_q, and let $\mathcal{M}$ be an ω-categorical structure with domain V and such that one of the following holds:*

(i) $\mathrm{PGL}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma L}(V)$

(ii) $\mathrm{PSp}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma Sp}(V)$

(iii) $\mathrm{PU}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma U}(V)$

(iv) $\mathrm{PO}(V) \leq \mathrm{Aut}(\mathcal{M}) \leq \mathrm{P\Gamma O}(V)$

Then $\mathcal{M}$ has a weak $\forall\exists$ interpretation.

The proof is contained in 2.2, 2.4, 3.11, 3.12, 4.6, 4.7, 4.12, 4.21, 4.25, 4.27.

It should be mentioned that reconstruction results were already known for the above permutation groups, since they have the small index property [**7**]. What is new is that these structures have weak $\forall\exists$ interpretations, and, in the case of spaces with forms, it may be new even that they are parameter-interpretable in their automorphism groups. In [**15**], Tolstykh uses techniques similar to those in Section 3 below to prove that the projective space $\mathrm{PG}(V)$ with the incidence relation is interpretable without parameters in all of $\Gamma\mathrm{L}(V), \mathrm{PGL}(V), \mathrm{P\Gamma L}(V)$ and

GL(V). Tolstykh's main result that PG(V) is interpretable in PGL(V) is based on the definability in PGL(V) of pairs of projective images of extremal involutions having a unique mutual subspace of V. Our results differ from Tolstykh's in that we provide an interpretation with parameters which yields reconstruction up to bi-interpretability, and we do so also in the case of spaces with forms.

The paper contains several results which we hope will have other applications. In particular, Proposition 2.2 and Lemma 2.4 give a method for lifting weak $\forall\exists$ interpretations from closed normal subgroups.

2. *Preliminaries*

Let V be a countably infinite dimensional vector space over a finite field F. Then V is determined up to isomorphism by its dimension, so it is an ω-categorical structure, and so are the symplectic, unitary and orthogonal spaces (V, β, Q) (where β is a sesquilinear form and Q the associated quadratic form in the orthogonal case). The projective spaces corresponding to these spaces are also ω-categorical. We shall produce weak $\forall\exists$ interpretations for various groups acting on PG(V) and on projective spaces with forms. We concentrate on the reconstruction of the projective spaces, rather than the vector space itself, because reconstruction for V via a weak $\forall\exists$ interpretation cannot be obtained in general, as Lemma 2.1 will show. Below we take Aut(V) to be the general linear group GL(V). The following argument applies for any ω-categorical structure $\mathcal{M}$ such that Aut($\mathcal{M}$) is transitive with non-trivial centre.

LEMMA 2.1. *Let V be as above, and suppose $F \neq \mathbb{F}_2$. Then there is no weak $\forall\exists$ interpretation for $\langle \mathrm{GL}(V), V\rangle$.*

Proof. Assume for a contradiction that $\langle \tau, C, \phi\rangle$ is a weak $\forall\exists$ interpretation for $\langle \mathrm{GL}(V), V\rangle$, where $C = (g_1, \ldots, g_n)^{\mathrm{GL}(V)}$, so that

$$\tau : \langle \mathrm{GL}(V), C/E_\phi\rangle \cong \langle \mathrm{GL}(V), V\rangle.$$

Let $v = \tau((g_1, \ldots, g_n)/E)$.

Consider a central element $g \in Z(\mathrm{GL}(V))$, $g \neq \mathrm{id}_{GL(V)}$. So $g = \lambda \mathrm{id}_{GL(V)}$ for some $\lambda \in F \setminus \{0\}$, $\lambda \neq 1$. Then $((g_1, \ldots, g_n)/E)^g = (g_1, \ldots, g_n)/E$, yet $v^g = \lambda v \neq v$. So g fixes $(g_1, \ldots, g_n)/E$ but not $\tau((g_1, \ldots, g_n)/E)$, which is a contradiction. $\square$

The proof of Lemma 2.1 suggests that the problem with a weak $\forall\exists$ interpretation for $\langle GL(V), V\rangle$ is created by scalars, so it is natural to turn our attention to the projective space PG(V), whose domain is the set of one-dimensional subspaces of V. There are various closed groups acting on $PG(V)$. The most natural group to consider is PGL(V). We have $\mathrm{PGL}(V) \trianglelefteq \mathrm{P\Gamma L}(V)$, where $\mathrm{P\Gamma L}(V)$ is the group of projective semilinear transformations on V, which is also closed and hence the automorphism group of a structure with domain PG(V).

Recall that if $G \leq \mathrm{Sym}(\Omega)$ is a closed subgroup of the full symmetric group of a countable set Ω, we can impose a canonical structure $\mathcal{O}$ on Ω in a canonical language L, where L contains an n-ary relation symbol R_Δ for each orbit Δ of G on Ω^n. If G acts oligomorphically on Ω, as in our case, the canonical structure is the structure on Ω whose 0-definable relations are determined by the action of the automorphism group. Any structure on Ω with G as automorphism group has the same 0-definable relations as the canonical structure. We shall henceforth assume that the structures we are working with are the canonical ones, determined by the action of the groups considered, and thus we shall not specify the language.

Our aim is to obtain a weak $\forall\exists$ interpretation for all the structures living on $\mathrm{PG}(V)$ determined by those closed groups H acting on $\mathrm{PG}(V)$ such that $\mathrm{PGL}(V) \leq H \leq \mathrm{P\Gamma L}(V)$. These include $\mathrm{PGL}(V)$, $\mathrm{P\Gamma L}(V)$ and some intermediate subgroups of $\mathrm{P\Gamma L}(V)$ which form a normal series. Propositions 2.2 and Lemma 3.12 below will ensure that it suffices to find a weak $\forall\exists$ interpretation for the structure determined by $\mathrm{PGL}(V)$ in order to have interpretations for the whole range of structures between $\langle \mathrm{PGL}(V), \mathrm{PG}(V)\rangle$ and $\langle \mathrm{P\Gamma L}(V), \mathrm{PG}(V)\rangle$.

Our result on $\mathrm{PG}(V)$ rests on the definition of weak $\forall\exists$ interpretation generalized to conjugacy classes on tuples, mentioned in remark 1.4. Our weak $\forall\exists$ interpretation will be based on a conjugacy class on pairs of automorphisms, where elements in each pair share a unique common fixed point. Moreover we shall work with various closed subgroups of $\mathrm{Sym}(\mathrm{PG}(V))$, and for this we need the facts that follow.

Proposition 2.2. *Let $G = \mathrm{Aut}(\mathcal{M})$, $\mathcal{M}$ an ω-categorical structure, and let $H \lhd G$ be a closed subgroup which is oligomorphic and transitive on $\mathcal{M}$ and $\exists$ definable in G. Suppose $\langle H, \mathcal{M}\rangle$ has a weak $\forall\exists$ interpretation $\langle H, C/E\rangle$ where*

(i) *$C \subseteq H^n$ consists of n-tuples of automorphisms $\langle g_0, \dots, g_n\rangle$ having the same fixed space, that is, $\mathrm{fix}(g_0) = \mathrm{fix}(g_1) = \ldots = \mathrm{fix}(g_n)$;*

(ii) *the equivalence relation E on C is defined by an existential formula $\phi(x, y, \bar{h})$;*

(iii) *$\bar{g}E\bar{k}$ if and only if $\mathrm{fix}(g_i) = \mathrm{fix}(k_i)$ for $i = 0, \dots, n$;*

(iv) *the bijection τ takes $\langle g_0, \dots, g_n\rangle/E$ to $m \in \mathrm{fix}(g_0)$.*

Then $\langle G, \mathcal{M}\rangle$ has a weak $\forall\exists$ interpretation.

Proof. For ease of notation, we shall take $n = 2$. Let $C = \langle h_0, h_1\rangle^H$ be the conjugacy class involved and ϕ be the existential formula defining the equivalence relation E on C, so that

$$\tau : \langle H, \langle h_0, h_1\rangle^H/E\rangle \cong \langle H, \mathcal{M}\rangle.$$

Let $\hat{C} = \langle h_0, h_1\rangle^G$. By normality of H, $\hat{C} \subseteq H \times H$. We would like to define $\hat{E}$ on $\hat{C}$ so that there is an isomorphism

$$\hat{\tau} : \langle G, \hat{C}/\hat{E}\rangle \cong \langle G, \mathcal{M}\rangle.$$

The obvious choice is to identify elements of $\hat{C}$ which have the same fixed space in $\mathcal{M}$. We know by hypothesis that elements of C, hence of $\hat{C}$, have the same fixed points in their action on $\mathcal{M}$, hence the same happens in their action on C/E. So we can define $\hat{E}$ by identifying $\langle g_0, g_1\rangle, \langle k_0, k_1\rangle \in \hat{C}$ whenever their fixed points in the action on C/E are the same, that is

$$\langle g_0, g_1\rangle \hat{E} \langle k_0, k_1\rangle \quad \text{iff} \quad \forall \langle x_0, x_1\rangle \in C(\ (\langle x_0, x_1\rangle^{g_0} E \langle x_0, x_1\rangle \wedge \langle x_0, x_1\rangle^{g_1} E \langle x_0, x_1\rangle)$$
$$\leftrightarrow (\langle x_0, x_1\rangle^{k_0} E \langle x_0, x_1\rangle \wedge \langle x_0, x_1\rangle^{k_1} E \langle x_0, x_1\rangle)).$$

Note that if $\langle g_0, g_1\rangle, \langle k_0, k_1\rangle \in \hat{C}$, then $\langle x_0, x_1\rangle^{g_i}$ and $\langle x_0, x_1\rangle^{k_i}$ are in C, so $\hat{E}$ is defined. By its form, $\hat{E}$ is an equivalence relation in any group, and it is G-invariant. We claim further that $\hat{E}$ is $\forall\exists$ definable in G in the language of groups.

Let ψ be an $\exists$ formula defining H. Then C is also $\exists$ definable (with parameters

$h_0, h_1)$ via the formula

$$\chi(x_0, x_1, h_0, h_1) \equiv \exists y(\psi(y) \wedge \langle x_0, x_1 \rangle = \langle h_0, h_1 \rangle^y).$$

Let $\bar{x} = \langle x_0, x_1 \rangle$. Now it is easy to define $\hat{E}$ by

$$\hat{\phi}(\bar{x}, \bar{y}, h_0, h_1) \equiv \forall \bar{z}(\chi(\bar{z}) \to ((\bar{z}^{x_0} E \bar{z} \wedge \bar{z}^{x_1} E \bar{z}) \leftrightarrow (\bar{z}^{y_0} E \bar{z} \wedge \bar{z}^{y_1} E \bar{z})).$$

The fact that E is $\exists$ definable guarantees that $\hat{E}$ is $\forall\exists$ definable. □

Remark 2.3. We shall use Proposition 2.2 with $n = 1$ when treating spaces with forms. The contents of Lemma 3.12 below is similar and is needed for $\langle \mathrm{PGL}(V), \mathrm{PG}(V) \rangle$.

When choosing H to be $\exists$ definable in the hypotheses of the previous Proposition, we have in mind the case when H contains a **generic automorphism**, that is, an automorphism which lies in a comeagre conjugacy class. If so, the following definability result holds.

Lemma 2.4. *Let G be a Polish group, $H \lhd G$ contain an element h generic in H. Then H is $\exists$ definable in G.*

Proof. Let $C_H = h^H$ be the comeagre conjugacy class of h. First, as is well known, any element $k \in H$ can be written as the product of two generics, as C comeagre implies $C \cap kC \neq \emptyset$. Then choose $g_0 \in C \cap kC$, so that $g_0 = kg_1$ for some $g_1 \in C$. Then $k = g_0 g_1^{-1}$. Since $C = C^{-1}$, $g_0, g_1 \in C$ as required.

Consider now the conjugacy class C_G of h in G. Since $C_H \subseteq C_G$ every element of H is a product of two elements of C_G, and C_G is $\exists$ definable in G with parameter h. So we have $H \subseteq C_G C_G$. By normality of H, $C_G \subseteq H$ so we can define $H = C_G C_G$. □

3. *A weak* $\forall\exists$ *interpretation for* $\mathrm{PG}(V)$

We shall define a conjugacy class on pairs $\hat{C} \subset \mathrm{PGL}(V) \times \mathrm{PGL}(V)$ and an equivalence relation E on C, $\forall\exists$ definable in the language of groups with parameters such that

$$\langle \mathrm{PGL}(V), \mathrm{PG}(V) \rangle \cong \langle \mathrm{PGL}(V), \hat{C}/E \rangle$$

as permutation groups. Given the extension of Rubin's definition in [**13**] to weak $\forall\exists$ interpretations defined with a conjugacy class on a tuple, we shall obtain a weak $\forall\exists$ interpretation for $\mathrm{PGL}(V)$ acting on $\mathrm{PG}(V)$. By Lemma 3.12 below, a weak $\forall\exists$ interpretation for $\langle \mathrm{PGL}(V), \mathrm{PG}(V) \rangle$ suffices to cover all the structures on $\mathrm{PG}(V)$ determined by those closed groups H such that $\mathrm{PGL}(V) \trianglelefteq H \trianglelefteq \mathrm{P\Gamma L}(V)$. Since $\mathrm{P\Gamma L}(V)/\mathrm{PGL}(V)$ is a finite cyclic group, these groups are closely related.

3.1. *Transvections*

Definition 3.1. A **transvection** is $\tau \in \mathrm{GL}(V)$ such that there are a linear functional u in the dual space V' and a vector $d \in V \setminus \{0\}$ such that

– $d\tau = d$
– $x\tau = x + (xu)d$ for all $x \in V$

We shall write $\tau_{d,u}$ for the transvection above. We shall call $\langle d\rangle$ the **direction** of τ.

The linear functional u will define a hyperplane U of equation $xu = 0$ and τ fixes U pointwise. Also, $d\tau = d$, hence $d \in U$. Given a transvection τ, we shall indicate the direction of τ by d_τ, and the fixed hyperplane by U_τ. Note that different transvections might have the same direction and the same fixed hyperplane. In fact the following hold:

PROPOSITION 3.2. *Let λ be a scalar, u, u' nonzero linear functionals and d, d' nonzero vectors. Then*
(i) $\tau_{\lambda d,u} = \tau_{d,\lambda u}$
(ii) $\tau_{d,u} = \tau_{d',u'}$ *if and only if there is a nonzero scalar μ such that $d' = \mu d$ and $u' = \mu^{-1}u$.*

Proof. By direct calculation, using the formula defining a transvection.

We also recall the following facts about transvections (see [**3**] and [**12**]):

LEMMA 3.3. *If $g \in \mathrm{GL}(V)$ and $\tau_{d,u} \in \mathrm{GL}(V)$ is a transvection, $\tau^g_{d,u} = \tau_{d^g,g^{-1}u}$.*

[**3**] 2.4.3. □

PROPOSITION 3.4. *There is a conjugacy class T in $\mathrm{GL}(V)$ consisting of all the transvections.*

Proof. [**3**] 2.4.4. □

PROPOSITION 3.5. *Let τ and σ be nontrivial transvections in $\mathrm{GL}(V)$. Then $\tau\sigma$ is a transvection if and only if $U_\tau = U_\sigma$ or $\langle d_\tau\rangle = \langle d_\sigma\rangle$.*

Proof. [**12**] 1.17. □

LEMMA 3.6. *The mapping $\widehat{\ }: \mathrm{GL}(V) \to \mathrm{PGL}(V)$ which takes $g \in \mathrm{GL}(V)$ to the mapping $\hat{g}$ defined by*

$$\langle v\rangle\hat{g} := \langle vg\rangle$$

is a group homomorphism which is continuous and open.

Proof. It is easy to check that $\widehat{\ }$ is a group homomorphism. To prove that $\widehat{\ }$ is continuous, consider a basic open set in $\mathrm{PGL}(V)$, say

$$\hat{U} = \{\hat{g} \in \mathrm{PGL}(V) : \langle v_i\rangle^{\hat{g}} = \langle w_i\rangle, i = 1, \ldots, n\}.$$

The inverse image of $\hat{U}$ is

$$U = \bigcup_{\alpha_i,\beta_i \in F\setminus\{0\}} \{g \in \mathrm{GL}(V) : \alpha_i v_i^g = \beta_i w_i, i = 1, \ldots n\}$$

which is a union of open sets, hence it is open.

Consider $\phi \in \mathrm{Stab}_{\mathrm{PGL}(V)}(\langle v_1\rangle \ldots \langle v_n\rangle)$. By reordering the v_i if necessary, let $\{v_1, \ldots, v_m\}$ be a maximally independent subset of $\{v_1, \ldots, v_n\}$. Extend $\{v_1, \ldots, v_m\}$

to a basis $\mathcal{B} = \{v_1, \ldots, v_m, w_{m+1}, \ldots\}$ of V. Choose any $g \in \mathrm{GL}(V)$ such that $\hat{g} = \phi$, and consider $\mathcal{B}' = \{v_1, \ldots, v_m, w^g_{m+1}, w^g_{m+2}, \ldots\}$. Clearly $\mathcal{B}'$ is also a basis, so, by transitivity of $\mathrm{GL}(V)$ there is $h \in \mathrm{GL}(V)$ taking $\mathcal{B}$ to $\mathcal{B}'$, $\hat{h} = \phi$ and $h \in \mathrm{Stab}_{\mathrm{GL}(V)}(v_1, \ldots, v_n)$. Hence $\phi \in \widehat{\mathrm{Stab}_{\mathrm{GL}(V)}}(v_1 \ldots v_n)$. Then $\mathrm{Stab}_{\mathrm{PGL}(V)}(\langle v_1 \rangle \ldots \langle v_n \rangle) \subseteq \widehat{\mathrm{Stab}_{\mathrm{GL}(V)}}(v_1 \ldots v_n)$, and the image of a basic open set is again open. □

Definition 3.7. We define $\hat{\tau} \in \mathrm{PGL}(V)$ to be a **projective transvection** if it is the image under $\widehat{\ }$ of some transvection $\tau \in \mathrm{GL}(V)$.

Since $\widehat{\ }$ is a homomorphism, by 3.4 projective transvections form a complete conjugacy class $\widehat{T}$ in $\mathrm{PGL}(V)$.

Lemma 3.8.
(i) *The preimage of the projective transvection $\hat{\tau}$ under $\widehat{\ }$ contains all nonzero scalar multiples of τ and nothing else.*
(ii) *A scalar multiple of a transvection is not a transvection. In particular, if τ, σ are transvections then $\lambda\tau = \sigma \iff \lambda = 1$ and $\tau = \sigma$.*

Proof. [**12**] 1.15. □

Let $\hat{\tau} \in \mathrm{PGL}(V)$ be a projective transvection, so that there is $\tau \in T$ whose image under $\widehat{\ }$ is $\hat{\tau}$. Then such a τ is unique and we shall call it the transvection **associated with** $\hat{\tau}$. Hence to each projective transvection $\hat{\tau}$ there remain associated a unique fixed hyperplane $U_{\hat{\tau}}$ and a unique direction $\langle d_{\hat{\tau}} \rangle$, which are those of the associated transvection. In what follows we shall always assume that τ is the transvection associated with $\hat{\tau}$ and that U_τ, $\langle d_\tau \rangle$ are the corresponding hyperplane and direction. From these considerations it is easy to obtain a projective version of Proposition 3.5:

Proposition 3.9. *Let $\hat{\tau}, \hat{\sigma}$ be non trivial projective transvections. Then $\hat{\tau}\hat{\sigma}$ is a transvection if and only if $U_\tau = U_\sigma$ or $\langle d_\tau \rangle = \langle d_\sigma \rangle$.*

Proof. [**12**], 1.23 □

3.2. *The interpretation*

We shall select a conjugacy class of pairs of projective transvections $\hat{C} \subseteq \mathrm{PGL}(V) \times \mathrm{PGL}(V)$ so that transvections in the same pair have the same direction and different fixed hyperplane, and an equivalence relation E on $\hat{C}$ identifying pairs having the same direction.

Proposition 3.10. *Let $(\hat{\sigma}, \hat{\sigma}') \in \mathrm{PGL}(V) \times \mathrm{PGL}(V)$ be two transvections such that $\langle d_\sigma \rangle = \langle d_{\sigma'} \rangle$ and $U_\sigma \neq U_{\sigma'}$. Then for all $\langle d \rangle \in \mathrm{PG}(V)$ there are $\hat{g} \in \mathrm{PGL}(V)$ and a pair $(\hat{\tau}, \hat{\tau}')$ of transvections such that $(\hat{\sigma}, \hat{\sigma}')^{\hat{g}} = (\hat{\tau}, \hat{\tau}')$ and $\langle d \rangle = \langle d_\tau \rangle = \langle d_{\tau'} \rangle$, $U_\tau \neq U_{\tau'}$.*

Proof. Clearly, if $(\hat{\sigma}, \hat{\sigma}')$ is such that $\langle d_\sigma \rangle = \langle d_{\sigma'} \rangle$ and $U_\sigma \neq U_{\sigma'}$, and $\hat{g} \in \mathrm{PGL}(V)$ is such that $\hat{\sigma}^{\hat{g}} = \hat{\tau}$, $(\sigma')^g = \tau'$, then (by 3.3) $\langle d_\tau \rangle = \langle d_{\tau'} \rangle$ and $U_\tau \neq U_{\tau'}$. Since $\mathrm{GL}(V)$ is transitive on the points of $\mathrm{PG}(V)$, given any $\langle d \rangle \in \mathrm{PG}(V)$ we can find $g \in \mathrm{GL}(V)$, hence $\hat{g} \in \mathrm{PGL}(V)$, such that $\langle d_\sigma \rangle^{\hat{g}} = \langle d \rangle$. Then $(\hat{\sigma}, \hat{\sigma}')^{\hat{g}}$ will be our required pair. $\square$

This lemma ensures that all points of $\mathrm{PG}(V)$ are represented by at least a pair in $\hat{C} = \{(\hat{\sigma}, \hat{\sigma}')^{\hat{g}} : \hat{g} \in \mathrm{PGL}(V)\}$. We can now obtain an $\forall\exists$ formula in the language of groups which defines pairs of transvections representing the same point of $\mathrm{PG}(V)$.

Proposition 3.11. *Let $(\hat{\rho}, \hat{\rho}')$ and $(\hat{\sigma}, \hat{\sigma}')$ be in $\hat{C}$ as defined above. Then*

$$(\hat{\rho}, \hat{\rho}')E(\hat{\sigma}, \hat{\sigma}') \text{ iff } \langle d_\rho \rangle = \langle d_\sigma \rangle$$

is a conjugacy invariant equivalence relation on $\hat{C}$, $\exists$ definable with parameters in $\mathrm{PGL}(V)$. *Hence there is a weak $\forall\exists$ interpretation for* $\langle \mathrm{PGL}(V), \mathrm{PG}(V) \rangle$.

Proof. Suppose $(\hat{\sigma}, \hat{\sigma}')$ is in $\hat{C}$ (so $\langle d_\sigma \rangle = \langle d_{\sigma'} \rangle$ and $U_\sigma \neq U_{\sigma'}$). We claim that $\langle d_\sigma \rangle = \langle d_\rho \rangle$ if and only if the products $\hat{\sigma}\hat{\rho}$ and $\hat{\sigma}\hat{\rho}'$ are both projective transvections.
Clearly $\langle d_\sigma \rangle = \langle d_\rho \rangle$ implies that $\hat{\sigma}\hat{\rho}$ and $\hat{\sigma}\hat{\rho}'$ are projective transvections. To prove the converse, suppose for a contradiction that $\hat{\sigma}\hat{\rho}$ and $\hat{\sigma}\hat{\rho}'$ are projective transvections, yet $\langle d_\sigma \rangle \neq \langle d_\rho \rangle$ (hence also $\langle d_\sigma \rangle \neq \langle d_{\rho'} \rangle$). Then by 3.9 $\hat{\sigma}\hat{\rho}$ is a transvection if and only if $U_\sigma = U_\rho$. Likewise, $\hat{\sigma}\hat{\rho}'$ is a transvection if and only if $U_\sigma = U_{\rho'}$. But then $U_\rho = U_{\rho'}$, contradicting $(\hat{\rho}, \hat{\rho}') \in \hat{C}$. Hence the formula

$\phi(x, x', y, y') \equiv xy$ is a projective transvection and xy' is a projective transvection

defines the equivalence relation E in the language of groups. By 3.3 E is conjugacy invariant. Note that the property of being a projective transvection is definable with a single parameter (say $\hat{\sigma}$) by the existence of a conjugating element to $\hat{\sigma}$ (by 3.4), so ϕ is in fact

$$\exists w \exists z \ ((xy)^w = \hat{\sigma} \wedge (xy')^z = \hat{\sigma}),$$

which is an existential formula. It is then easy to see that

$\psi(x, x', y, y') \equiv \phi(x, x', y, y') \wedge$ '$\phi(x, x', y, y')$ defines an equivalence relation on $\hat{C}$'

is an $\forall\exists$ equivalence formula (with parameters $\hat{\sigma}, \hat{\sigma}'$ such that $(\hat{\sigma}, \hat{\sigma}') \in \hat{C}$).
Then $\langle \psi(x, x', y, y'), \hat{\sigma}, \hat{\sigma}', \tau \rangle$ is a weak $\forall\exists$ interpretation for $\langle \mathrm{PGL}(V), \mathrm{PG}(V) \rangle$, where $\tau : \hat{C}/E \to \mathrm{PG}(V)$ is defined by $\tau((\hat{\rho}, \hat{\rho}')/E) = \langle d_\rho \rangle$. $\square$

Lemma 3.12. *Let G be a closed group and such that* $\mathrm{PGL}(V) \leq G \leq \mathrm{P\Gamma L}(V)$. *Then $\langle G, \mathrm{PG}(V) \rangle$ has a weak $\forall\exists$ interpretation.*

Proof. Let $\hat{C} = (\hat{\sigma}, \hat{\sigma}')^{\mathrm{PGL}(V)}$ be the conjugacy class on pairs of transvections which gives the weak $\forall\exists$ interpretation of Proposition 3.11 above. Since $\mathrm{PGL}(V) \lhd G$, we have $\hat{C} \subseteq (\hat{\sigma}, \hat{\sigma}')^G \subseteq \mathrm{PGL}(V) \times \mathrm{PGL}(V)$. Hence $(\hat{\sigma}, \hat{\sigma}')^G$ is again made of pairs of transvections $(\hat{\rho}, \hat{\rho}')$ such that $\langle d_\rho \rangle = \langle d_{\rho'} \rangle$ but $U_\rho = U_{\rho'}$. Then we define $\hat{E}$ on $(\hat{\sigma}, \hat{\sigma}')^G$ with exactly the same formula as in 3.11, so that $(\hat{\rho}, \hat{\rho}')\hat{E}(\hat{\sigma}, \hat{\sigma}')$ iff $\langle d_\rho \rangle = \langle d_\sigma \rangle$. $\square$

gle H, M$\rangle$.

4. *Spaces with forms*

Let V be a vector space as above, and suppose $\sigma \in \mathrm{Aut}(F)$. Let us recall some basic definitions and notation. A **sesquilinear form** on V is a map $\beta : V \times V \to F$ such that for all $u_i, v_i \in V$, $a, b \in F$

(1) $\beta(u_1 + u_2, v) = \beta(u_1, v) + \beta(u_2, v)$
(2) $\beta(u, v_1 + v_2) = \beta(u, v_1) + \beta(u, v_2)$
(3) $\beta(au, bv) = ab^\sigma \beta(u, v)$

The form β is said to be

– **alternating** if $\sigma = 1 \in \mathrm{Aut}(F)$ and $\beta(v, v) = 0$ for all v in V;
– **symmetric** if $\sigma = 1 \in \mathrm{Aut}(F)$ and $\beta(u, v) = \beta(v, u)$ for all u, v in V;
– **hermitian** if $\sigma \neq 1, \sigma^2 = 1 \in \mathrm{Aut}(F)$ and $\beta(u, v) = \sigma(\beta(v, u))$ for all u, v in V.

If β is alternating then $\beta(u, v) = -\beta(v, u)$ for all $u, v \in V$.

If X is a subspace of V we define $X^\perp := \{u \in V : \forall x \in X \ \beta(u, x) = 0\}$. Note that $X^\perp \leq V$. The **radical** of V is $V^\perp$. If $U \leq V$, $\mathrm{Rad}(U) = U \cap U^\perp$. The form β is said to be **nondegenerate** if $\mathrm{Rad}(V) = \{0\}$.

A **quadratic form** on V is a function $Q : V \to F$ such that

$$Q(av) = a^2 Q(v) \text{ for all } a \in F, v \in V, \text{ and}$$

$$\beta(u, v) := Q(u + v) - Q(u) - Q(v)$$

is a bilinear form (i.e. sesquilinear with $\sigma = 1$). Then β is symmetric, and it is called the bilinear form **associated with** Q. By the definition, Q determines β and $\beta(u, u) = 2Q(u)$. If $\mathrm{char}(F) = 2$, we get that $\beta(u, u) = 0$ for all $u \in V$.

The forms defined above give rise to three kinds of spaces:

– the symplectic space (V, β), where β is alternating nondegenerate;
– the orthogonal space (V, β, Q), where β is symmetric nondegenerate;
– the unitary space (V, β), where β is hermitian nondegenerate.

If V is countably infinite dimensional and F is finite then each form is unique, so the space (V, β) is an ω-categorical structure. Unlike the vector space case, categoricity does not hold in uncountable dimension. Our convention about adopting the canonical language will hold for spaces with forms.

Definition 4.1. If (V_1, β_1, Q_1) and (V_2, β_2, Q_2) are F vector spaces as above (both symplectic or both orthogonal or both unitary) then $f : V_1 \to V_2$ is a **linear isometry** if f is linear and for all $u, v \in V_1$

$$\beta_2(uf, vf) = \beta_1(u, v) \text{ and } Q_2(vf) = Q_1(v).$$

We shall denote the isometry group of the space (V, β, Q) by $\mathrm{O}(V, \beta, Q)$. This notation covers the symmetric, the unitary and the orthogonal groups. We shall write $\mathrm{Sp}(V)$ and $\mathrm{PSp}(V)$ for the symplectic and projective symplectic groups respectively. $\mathrm{O}(V)$, $\mathrm{PO}(V)$, $\mathrm{U}(V)$ and $\mathrm{PU}(V)$ will denote the orthogonal, projective orthogonal, unitary and projective unitary groups respectively.

We now need more definitions:

Definition 4.2.

(i) A non-zero vector $v \in V$ is **isotropic** if $\beta(v, v) = 0$;

(ii) a subspace $W \subseteq V$ is **totally isotropic** if $W \subseteq W^{\perp}$;
(iii) a non-zero vector v is **singular** if $Q(v) = 0$ (note that in odd characteristic a vector is singular if and only if it is isotropic);
(iv) $W \subseteq V$ is **totally singular** if $Q(w) = 0$ for all $w \in W$;
(v) $W \subseteq V$ is **non-degenerate** if $W \cap W^{\perp} = \{0\}$;
(vi) if $V = U \oplus V$ and $\beta(u, v) = 0$ for all $u \in U$, $v \in V$, we say that V is an **orthogonal direct sum** of U and V, and we write $U \perp V$.

DEFINITION 4.3. A pair of vectors u, v such that u, v are both isotropic and $\beta(u, v) = 1$ is called a **hyperbolic pair**, and the line $\langle u, v\rangle$ in $\mathrm{PG}(V)$ is a **hyperbolic line**. In the presence of a quadratic form Q, we also require that $Q(u) = Q(v) = 0$.

We shall now state Witt's theorem, which is a major result concerning spaces with forms and which we shall repeatedly need:

THEOREM 4.4 Witt. *Let V be a nondegenerate symplectic, orthogonal or unitary space, where V has dimension ω over the finite field F, and let $U \leq V$ be a finite dimensional subspace. Suppose that $g : U \to V$ is a linear isometry. Then there is a linear isometry $h : V \to V$ such that $ug = uh$ for all $u \in U$.*

Proof. [**14**], 7.4. □

In particular any isomorphism between two non degenerate subspaces of V can be extended to a full isomorphism.

4.1. *Generics in* $\mathrm{PO}(V, \beta, Q)$

In this section we shall establish some facts about isometry groups which are needed later for finding weak $\forall\exists$ interpretations for spaces with forms.

We shall think of (V, β, Q) as the Fraïssé limit of finite dimensional spaces having a hyperbolic basis. We refer the reader to the literature for proofs that, when the underlying field is finite, the even dimensional vector space U can be equipped with an orthogonal and unitary form admitting a hyperbolic basis. We shall show that $\mathrm{O}(V, \beta, Q)$ contains a generic automorphism, so that 2.2 applies (by 2.4).

The following is well known, and it is central to the fact that vector spaces over finite fields with non degenerate bilinear forms are smoothly approximable (cf. [**4**]).

LEMMA 4.5. *Let V be a countably infinite dimensional vector space over a finite field F. Let β be a nondegenerate alternating, symmetric or hermitian form on V. Suppose that $U \leq V$ is a finite dimensional subspace. Then there is a finite dimensional $\bar{U} \leq V$ such that $\bar{U}$ is nondegenerate and $U \leq \bar{U}$.*

We can now prove the existence of a generic automorphism.

LEMMA 4.6. *The isometry group $\mathrm{O}(V, \beta, Q)$ contains a generic automorphism.*

Proof. Let $G = \mathrm{O}(V, \beta, Q)$, and let $\mathbb{P} = \{p : V \to V$ s.t. p is a partial finite isometry}. By theorem 2.1 in [**16**], it suffices to find a cofinal subset of $\mathbb{P}$ which satisfies the amalgamation property and the joint embedding property.

Consider the set

$$\mathbb{E} := \{p \in \mathbb{P} : \mathrm{dom}(p) \text{ has a hyperbolic basis}\}.$$

It is clear that $\mathbb{E}$ is closed under conjugacy. For cofinality, let $p \in \mathbb{P}$ and consider $U = \langle \mathrm{dom}(p) \cup \mathrm{ran}(p)\rangle$. By 4.5, there is $\bar{U} \supseteq U$ such that $\bar{U}$ is nondegenerate. Since we think of V as the Fraïssé limit of finite dimensional spaces having a hyperbolic basis, we may assume that $\bar{U}$ has such a basis. By Witt's thorem we can extend p to an isometry $\bar{p}$ of V. Then $\bar{p}|_{\bar{U}} \in \mathbb{E}$ and $p \subseteq \bar{p}|_{\bar{U}}$.

For the amalgamation property, suppose that $h, g_1, g_2 \in \mathbb{E}$ and let U, V_1, V_2 be $\mathrm{dom}(h), \mathrm{dom}(g_1)$ and $\mathrm{dom}(g_2)$ respectively. Suppose further that $\alpha_i : \langle U, h\rangle \to \langle V_i, g_i\rangle$, $i = 1, 2$, are embeddings in $\mathbb{E}$. We want to find $g \in \mathbb{E}$, with $\mathrm{dom}(g) = W$, and $\gamma_i : \langle V_i, g_i\rangle \to \langle W, g\rangle$, $\gamma_i \in \mathbb{E}$, such that $\alpha_1\gamma_1 = \alpha_2\gamma_2$. By identifying U with $\alpha_i(U)$ we may assume that $\alpha_1 = \mathrm{id}$ and $\alpha_2 = \mathrm{id}$, so that $\langle U, h\rangle \subseteq \langle V_i, f_1\rangle$, $i = 1, 2$. Choose a hyperbolic basis $\mathcal{B} = \{e_1, f_1, \ldots, e_n, f_n\}$ for U, where each pair (e_i, f_i) is hyperbolic. Extend $\mathcal{B}$ to hyperbolic bases $\mathcal{B}_1 = \mathcal{B} \cup \{e_{n+1}, f_{n+1}, \ldots, e_r, f_r\}$ for V_1 and $\mathcal{B}_2 = \mathcal{B} \cup \{e'_{n+1}, f'_{n+1}, \ldots, e'_s, f'_s\}$ for V_2. Let $W = \langle \mathcal{B}_1 \cup \mathcal{B}_2\rangle$, $g = g_1 \cup g_2$ and define $\beta(e_i, e'_j) = \beta(f_i, f'_j) = \beta(e_i, f'_j) = \beta(e'_j, f_i) = 0$ for $i = n+1, \ldots, r$ and $j = n+1, \ldots, s$. It is easy to check that g respects β on this basis, hence $g \in O(W, \beta, Q)$ is the required extension of g_1, g_2.

The joint embedding property is proved similarly. □

Proposition 4.7. *Let f be a generic automorphism of $G \leq \mathrm{O}(V, \beta, Q)$. Then the projective image $\hat{f}$ is generic in $\hat{G}$.*

Proof. Let $C = g^G$ be the comeagre conjugacy class of g. Let $Z = Z(G) \leq \{\alpha I : \alpha \in F, I = \mathrm{id}_G\}$ be the centre of G.

First, we claim that for any $z \in Z$, $zC = C$. Indeed, clearly zC is comeagre (as translation by z is a homeomorphism of G), and $zC = \{zf^{-1}gf : f \in G\} = \{f^{-1}zgf : f \in G\} = (zg)^G$. Hence zC is a conjugacy class. As there is a unique comeagre conjugacy class, $zC = C$. Hence $ZC := \bigcup_{z\in Z} zC = C$.

It follows that if $C \supseteq \bigcap_{i\in\omega} D_i$, where each D_i is dense and open, then $C \supseteq \bigcap_{i\in\omega} ZD_i$, and each ZD_i is also dense and open. By 3.6, the sets $\widehat{ZD_i}$ are dense and open.

We argue that $\widehat{\bigcap_{i\in\omega} ZD_i} = \bigcap_{i\in\omega} \widehat{ZD_i}$. For $\subseteq$, if $x \in \widehat{\bigcap_{i\in\omega} ZD_i}$, then there is $h \in \bigcap_{i\in\omega} ZD_i$ with $x = \hat{h}$. But now $h \in ZD_i$ for all i, so $x \in \widehat{ZD_i}$ for all i, so $x \in \bigcap_{i\in\omega} \widehat{ZD_i}$.

For the reverse inclusion, suppose $x \in \bigcap_{i\in\omega} \widehat{ZD_i}$, with $x = \hat{h}$. Then for all i, $x \in \widehat{ZD_i}$, so for all i, $h \in ZD_i$. Hence $h \in \bigcap_{i\in\omega} ZD_i$, so $x \in \widehat{\bigcap_{i\in\omega} ZD_i}$.

So $\hat{C} \supseteq \widehat{\bigcap_{i\in\omega} ZD_i} = \bigcap_{i\in\omega} \widehat{ZD_i}$. Hence $\hat{C}$ contains a countable intersection of dense open sets, so it is comeagre, i.e. $\hat{g}$ is a generic. □

Hence $\mathrm{PSp}(V)$, $\mathrm{PU}(V)$ and $\mathrm{PO}(V)$ all contain a generic automorphism. By 2.4, this means that they are $\exists$ definable in $\mathrm{P\Gamma Sp}(V)$, $\mathrm{P\Gamma U}(V)$ and $\mathrm{P\Gamma O}(V)$ respectively. Since they are also normal in these groups, they satisfy the hypothesis of 2.2, and hence weak $\forall\exists$ interpretations based on existential formulae for each of them will suffice for reconstructing all the structures on $\mathrm{PG}(V)$ induced by groups respecting forms.

REMARK 4.8. Propositions 4.6 and 4.7 are implicit in [**5**]. Theorem 4.1 in [**5**] states that for any affine cover M there is n such that the set Λ_n of all finite n-saturated envelopes in M forms an amalgamation base. It is easy to see (from the definition of an amalgamation base) that the set $\mathcal{P}$ of all pairs (A, α), with $A \in \Lambda_n$ and $\alpha \in Sym(A)$ extendible to an automorphism of M, forms a cofinal subset of all finite partial automorphisms satisfying the properties of Theorem 2.1 in [**16**]. Thus 4.6 and 4.7 also follow from [**5**].

4.2. *The interpretation for* $\mathrm{Sp}(V)$

The following facts will yield a weak $\forall\exists$ interpretation for $\mathrm{PSp}(V)$ acting on $\mathrm{PG}(V)$:

PROPOSITION 4.9. $\mathrm{Sp}(V)$ *is transitive on the points of* $\mathrm{PG}(V)$.

Proof. This is a consequence of Witt's theorem. □

LEMMA 4.10. *If* $\tau \in \mathrm{Sp}(V)$ *is a transvection then* $U_\tau = d^\perp$.

Proof. Let $\tau = \tau_{d,u}$ where $u \in V^*$. Then:

$$\begin{aligned} \tau_{d,u} \in \mathrm{Sp}(V) &\iff \forall v, w \in V \ \beta(v,w) = \beta(v\tau_{d,u}, w\tau_{d,u}) \\ &= \beta(v + (v)ud, w + (w)ud) \\ &= \beta(v,w) + (v)u\beta(d,w) + (w)u\beta(v,d) \end{aligned}$$

Hence we need $(v)u\beta(d,w) + (w)u\beta(v,d) = 0$ for all $v, w \in V$. We can choose $v \in V$ with $\beta(d,v) = 1$. Then for all w we have $(w)u = (v)u\beta(d,w)$, that is $\ker(u) = u^\perp$. □

PROPOSITION 4.11. *There is a conjugacy class* $T = \tau_{d,u}^{\mathrm{Sp}(V)}$ *in* $\mathrm{Sp}(V)$ *consisting of symplectic transvections. Moreover, for all* $\langle v\rangle \in \mathrm{PG}(V)$, *there is* $\tau_{d',u'} \in T$ *with* $\langle d'\rangle = \langle v\rangle$.

Proof. First note that the conjugate of a symplectic transvection is a symplectic transvection: let $\tau_{d,u} \in \mathrm{Sp}(V)$ be a transvection, and let $g \in \mathrm{Sp}(V)$. Then, by 3.3, $\tau_{d,u}^g = \tau_{d^g, g^{-1}u}$. Since $\ker(u) = d^\perp$, we have that $(\ker(u))^g = (d^\perp)^g$. But $(\ker(u))^g = \ker(ug^{-1})$ and $(d^\perp)^g = (d^g)^\perp$, so $\tau_{d^g, g^{-1}u}$ is a symplectic transvection as required. The second claim then follows because $\mathrm{Sp}(V)$ is transitive on the points of $\mathrm{PG}(V)$. □

Proposition 4.11 ensures that if we work with $\hat{C} = \hat{\tau}_{d,u}^{\mathrm{PSp}(V)}$, where $\hat{\tau}_{d,u}$ is a projective symplectic transvection, each point in $\mathrm{PG}(V)$ will be represented by at least one element of $\hat{C}$.

Lemma 4.10 enables us to find a simpler interpretation for $\mathrm{PSp}(V)$ than the one for $\mathrm{PG}(V)$: since the direction of a transvection determines its fixed hyperplane uniquely, we can use Proposition 3.9 to identify those symplectic transvections that fix the same direction:

PROPOSITION 4.12. *Let $\hat{\tau}, \hat{\sigma}$ be projective symplectic transvections. Then:*

$$\hat{\tau}\hat{\sigma} \text{ is a projective symplectic transvection} \iff \langle d_\tau \rangle = \langle d_\sigma \rangle$$

Proof. This is a direct consequence of 3.9 and 4.10. □

It follows that the relation "having the same direction" on the conjugacy class of projective symplectic transvections is indeed an $\exists$ definable equivalence relation in the language of groups.

4.3. *A reconstruction result for* PU(V) *and* PO(V)

Our reconstruction results for the unitary and orthogonal spaces will involve selecting a suitable subset of V on which U(V) and O(V) are closed automorphism groups, and extending the interpretation to the full domain.

FACT 4.13. *The unitary space (V, β) has a basis of isotropic vectors. Moreover,* PU(V) *is transitive on the set of isotropic points of* PG(V).

Proof. [**14**] pp. 116–117 and Theorem 10.12. □

FACT 4.14. *There is an orbit P of the orthogonal group* O(V) *on (V, Q) which consists of nonsingular vectors and contains a basis for V.*

Proof. It is known that the orthogonal group O(V) is irreducible in its natural action on V, so any orbit spans V. In particular if $v \in V$ is nonsingular, then $\{v^g : g \in \mathrm{O}(V)\}$ consists of nonsingular vectors and it contains a basis, as required. □

We now prove that PO(V) acting on an orbit $\hat{P}$ of nonsingular points (resp. PU(V) acting the set $\hat{P}$ of isotropic points) is closed, and that $\mathrm{PG}(V) = \mathrm{dcl}(\hat{P})$. We shall use the following fact.

LEMMA 4.15. *Let $\mathcal{M}$ be a first order structure, W a set, and $\pi : \mathcal{M} \to W$ be a finite-to-one surjection whose fibres form an* Aut($\mathcal{M}$)*-invariant partition of $\mathcal{M}$. Let $\mu : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Sym}(W)$ be the map defined by $w\mu(g) = ((w)\pi^{-1}g)\pi$ for all $g \in \mathrm{Aut}(C)$ and $w \in W$. Then μ maps closed subgroups of* Aut($\mathcal{M}$) *to closed subgroups of* Sym(W).

Proof. [**8**], 1.4.2. □

PROPOSITION 4.16. *Let $\mathcal{M}$ be a structure, $G = \mathrm{Aut}(\mathcal{M})$ and $P \subseteq \mathcal{M}$ be a G-invariant subset such that $\mathcal{M} = \mathrm{dcl}(P)$. Then G is closed on P.*

Proof. Suppose that $g \in G$. Then, since $P^g = P$ and g is a bijection on $\mathcal{M}$, g is also a bijection on P.

Recall that G is closed in Sym(P) if and only if the following holds: if $g \in \mathrm{Sym}(P)$

is such that for all $\bar{p} \in P^n$ there is $h \in G$ such that $\bar{p}^h = \bar{p}^g$, then $g \in G$. So let $g \in \mathrm{Sym}(P)$ be as in the hypothesis, i.e. g behaves like an element of G on each finite tuple in P. We want to show that $g \in G$.

Extend g to $g' \in \mathrm{Sym}(\mathcal{M})$ as follows: for $m \in \mathcal{M}$, let $m \in \mathrm{dcl}(\bar{p})$, $\bar{p} \in P^k$, be defined by the formula $\phi(x, \bar{p})$. Choose $h \in G$ agreeing with g on $\bar{p}$, and extend g to g' defined by

$$m^{g'} := \phi(\mathcal{M}, \bar{p}^h).$$

Then g' is well-defined: if $m = \phi(\mathcal{M}, \bar{p})$ and $m = \psi(\mathcal{M}, \bar{q})$, then $\phi(\mathcal{M}, \bar{p}) = \psi(\mathcal{M}, \bar{q})$ implies that $\phi(\mathcal{M}, \bar{p}^h) = \psi(\mathcal{M}, \bar{q}^h)$. It is easy to see that g' is independent of the choice of h.

Now let $\bar{m} \in \mathcal{M}^n$, and let ψ be any n-formula. Suppose $\{m_i\} = \phi_i(\mathcal{M}, \bar{p}^i)$ for $i = 1, \ldots, n$. For each $i = 1, \ldots, n$ there is a 0-definable partial function f_i such that $m_i = f_i(\bar{p}^i)$. Then

$$\begin{aligned} \mathcal{M} \models \psi(\bar{m}) &\iff \mathcal{M} \models \psi(f_1(\bar{p}^1), \ldots, f_n(\bar{p}^n)) \\ &\iff \mathcal{M} \models \psi(f_1((\bar{p}^1)^h), \ldots, f_n((\bar{p}^n)^h)) \\ &\iff \mathcal{M} \models \psi(\bar{m}^{g'}). \end{aligned}$$

Hence $g' \in \mathrm{Aut}(\mathcal{M})$, as required. □

Proposition 4.17. *Let* $(\mathrm{PG}(V), \beta, Q)$ *be the projective unitary (resp. orthogonal) space, and* $G = \mathrm{PU}(V)$ *(resp.* $\mathrm{PO}(V)$*). Let* $\hat{P}$ *be the set of isotropic (resp. an orbit of nonsingular) 1-dimensional subspaces, and* $\mathcal{O}$ *be an orbit of* G *on* $(\mathrm{PG}(V), \beta, Q)$*. Then* $\mathcal{O} \subseteq \mathrm{dcl}(\hat{P})$*. It follows that* G *is faithful on* $\hat{P}$*.*

Proof. Let $\mathcal{O}$ be as in the statement. We know that the pre-image P of $\hat{P}$ under $\hat{}$ contains a basis for V, so every $v \in V$ is a linear combination of vectors in P. Let $\mathcal{O} = \langle v \rangle^{O(V,\beta,Q)}$, and suppose that $\langle v \rangle = \langle \alpha_1 v_1 + \ldots + \alpha_r v_r \rangle$, $\alpha_i \in F$, $v_i \in P$. If $f \in G$ fixes $\langle v_1 \rangle, \ldots, \langle v_r \rangle$, then $v_1, \ldots v_r$ have finitely many translates in V, hence $v_1 + \ldots + v_r$ has finitely many translates. So $\langle v \rangle \in \mathrm{acl}(\langle v_1 \rangle, \ldots, \langle v_r \rangle)$. So we have that $\mathcal{O} \subseteq \mathrm{acl}(\hat{P})$.

Suppose for a contradiction that there is $\langle v \rangle \in \mathcal{O}$ such that $\langle v \rangle \notin \mathrm{dcl}(\hat{P})$. Then, by a König's Lemma argument, there is $g \in G_{\hat{P}}$ such that $\langle v \rangle g \neq \langle v \rangle$. But then $G_{\hat{P}}$ is normal in G, closed and nontrivial, since it contains g. But, since Theorem 1 in [**9**] implies that G has no proper non trivial closed normal subgroups, this is a contradiction.

It follows that if $g \in G_{\hat{P}}$, then $\langle v \rangle g = \langle v \rangle$ for all $\langle v \rangle \in \mathrm{PG}(V)$, so $g = \mathrm{id}$, i.e. G is faithful. □

Corollary 4.18. *Let* P *be the set of isotropic vectors in the unitary space* (V, β)*, resp. an orbit of nonsingular vectors in the orthogonal space* (V, O)*. Then* $G = \mathrm{O}(V)$ *(resp.* $G = \mathrm{U}(V)$*) is closed in its action on* P*. It follows that the projective image* $\hat{G}$ *of* G *is closed in its action on* $\hat{P} := \{\langle v \rangle : v \in P\}$*.*

Proof. By 4.16 and 4.17, G is closed on P. By 4.15 with $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M} \rangle = \langle G, P \rangle$ and $W = \hat{P}$, the projective image $\hat{G}$ of G is closed in its action on $\hat{P}$. □

Hence $\mathrm{PO}(V)$ and $\mathrm{PU}(V)$ induce the automorphism group of a structure on an orbit of nonsingular 1-subspaces and on the set of isotropic 1-subspaces respectively. We shall start by looking for weak $\forall\exists$ interpretations for the structures $\langle \mathrm{PO}(V,\beta,Q), \hat{P}\rangle$ and later extend our results to $\langle \mathrm{PO}(V,\beta,Q), \mathrm{PG}(V)\rangle$.

FACT 4.19. *Suppose $\tau_{d,u}$ is a transvection in* $\mathrm{GL}(V)$*. Then $\tau_{d,u} \in \mathrm{U}(V)$ if and only if it is of the form*

$$v\tau = v + a\beta(v,d)d$$

where d is isotropic and $a \in F$ satisfies $a + a^\sigma = 0$. In particular, for each isotropic vector d there is a unitary transvection having direction $\langle d\rangle$.

Proof. [**14**], pp. 118–119. □

Projective unitary transvections are defined in the usual way. Note that here, as in the symplectic case, for a transvection $\tau_{d,u}$ we have $ker(u) = \langle d\rangle^\perp = d^\perp$, so our weak $\forall\exists$ interpretation for $\langle \mathrm{PU}(V), \hat{P}\rangle$ is based on the same formula as we used in the symplectic case.

PROPOSITION 4.20. *There is a conjugacy class $T = \hat{\tau}_{d,u}^{\mathrm{PU}(V)}$ in* $\mathrm{PU}(V)$ *such that for all isotropic $\langle v\rangle \in \mathrm{PG}(V)$, there is $\tau_{d',u'} \in T$ with $\langle d'\rangle = \langle v\rangle$.*

Proof. The proof follows from 4.19. □

PROPOSITION 4.21. *Let $\hat{\tau}$, $\hat{\sigma}$ be projective unitary transvections. Then*

$$\hat{\sigma}\hat{\tau} \text{ is a unitary projective transvection} \iff \langle d_\tau\rangle = \langle d_\sigma\rangle.$$

Proof. This is a consequence of the fact that for $\tau_{d,u}$ $\ker(u) = \langle d\rangle^\perp = d^\perp$ and of 3.9. □

The reconstruction result for the orthogonal space is very similar to the unitary case, except that when $\mathrm{char}(F) \neq 2$ there are no transvections in $\mathrm{O}(V)$ so we use reflections instead, and we need a basis of nonsingular, rather than isotropic, vectors. Let us deal with the characteristic 2 case first:

LEMMA 4.22. *If* $\mathrm{char}(F) = 2$*, the following hold:*
(i) *the orthogonal space (V,Q) contains a transvection τ;*
(ii) *$v\tau = v + Q(v)^{-1}\beta(v,u)u$ for a nonsingular vector u;*
(iii) *each nonsingular point in* $\mathrm{PG}(V)$ *is the centre of a unique transvection.*

Proof. [**1**], 22.3. □

So the even characteristic case is treated like the unitary case, except that, by virtue of 4.22 2. above, there is no need to quotient the conjugacy class of orthogonal transvections by an equivalence relation. For the general case, we need to define *reflections.*

DEFINITION 4.23. A **reflection** in $\mathrm{O}(V,Q)$ is a map of the form

$$\tau_u(v) = v - Q(u)^{-1}\beta(v,u)u$$

where u is a nonsingular vector. We call $\langle u\rangle$ the **centre** of τ_u.

Note that τ_u fixes $\langle u\rangle^{\perp}$. Moreover, for every nonsingular vector $u \in (V, Q)$ there is a unique reflection with centre $\langle u\rangle$:

$$\begin{aligned} v\tau_{\lambda u} &= v - Q(\lambda u)^{-1}\beta(v, \lambda u)\lambda u \\ &= v - \frac{Q(u)^{-1}}{\lambda^2}\lambda^2\beta(v, u)u \\ &= v\tau_u. \end{aligned}$$

DEFINITION 4.24. A **projective reflection** is an element of $\mathrm{PO}(V, Q)$ of the form $\hat{\tau}_u$ where τ_u is a reflection.

It follows easily from the above that for every nonsingular point of $\mathrm{PG}(V)$ there is a unique projective reflection with centre $\langle u\rangle$.

PROPOSITION 4.25. *For each orbit P of $\mathrm{O}(V)$ consisting of nonsingular vectors there is a conjugacy class $C \subseteq \mathrm{O}(V)$ consisting of reflections such that for all $v \in P$ there is a unique reflection in C having centre $\langle v\rangle$. It follows that there is a bijection between the conjugacy class $\hat{C} \subseteq \mathrm{PO}(V)$ and the orbit $\hat{P}$ such that $\langle \mathrm{PO}(V), \hat{P}\rangle \cong \langle \mathrm{PO}(V), \hat{C}\rangle$.*

Proof. Let $\tau_u \in \mathrm{O}(V, Q)$ be a reflection. Then

$$\begin{aligned} (v)g^{-1}\tau_u g &= (vg^{-1} - Q(u)^{-1}\beta(vg^{-1}, u)u)g \\ &= v - Q(u)^{-1}\beta(vg^{-1}, u)ug \\ &= v - Q(ug)^{-1}\beta(v, ug)ug \\ &= v\tau_{ug}. \end{aligned}$$

So the conjugate by $g \in \mathrm{O}(V)$ of a reflection with centre u is a reflection of centre u^g. Since $\mathrm{O}(V)$ is transitive on the orbit P, and by the remark following 4.24, the claim follows. □

The facts above yield a weak $\forall\exists$ interpretation for $\mathrm{PO}(V)$ acting on an orbit $\hat{P}$ of nonsingular points of $\mathrm{PG}(V)$. It is clear that in this case we do not need to find an equivalence relation on the conjugacy class considered, since there is naturally a bijection with the orbit $\hat{P}$.

So far we have obtained weak $\forall\exists$ interpretations for $\langle \mathrm{PU}(V), \hat{P}\rangle$, where $\hat{P}$ is the set of isotropic points in the projective unitary space $(\mathrm{PG}(V), \beta)$, and for $\langle \mathrm{PO}(V), \hat{P}\rangle$, where $\hat{P}$ is an orbit of nonsingular points in the orthogonal projective space $(\mathrm{PG}(V), Q)$. By 4.17, this gives a generalised weak $\forall\exists$ interpretation for $\langle \mathrm{PO}(V), \mathrm{PG}(V)\rangle$ and $\langle \mathrm{PU}(V), \mathrm{PG}(V)\rangle$.

Proposition 4.17 gives a weak $\forall\exists$ interpretation in the sense of 1.4 for $\mathrm{PO}(V)$ and $\mathrm{PU}(V)$ acting on $\mathrm{PG}(V)$. In order to lift these interpretations to $\mathrm{P\Gamma U}(V)$ and $\mathrm{P\Gamma O}(V)$ and to the intermediate closed subgroups, we prove the following extension of Proposition 4.17.

PROPOSITION 4.26. *Let G such that $\mathrm{PU}(V) \leq G \leq \mathrm{P\Gamma U}(V)$ (resp. $\mathrm{PO}(V) \leq G \leq \mathrm{P\Gamma O}(V)$) be a closed group on the set $\hat{P}$ of isotropic (resp. on an orbit of* *nonsingular) 1-dimensional subspaces of V. Let $\mathcal{O}$ be an orbit of G on* $\mathrm{PG}(V)$. *Then $\mathcal{O} \subseteq \mathrm{dcl}(\hat{P})$. It follows that G is faithful on $\hat{P}$.*

Proof. For ease of notation, we shall state the argument for $\mathrm{PU}(V) \leq G \leq \mathrm{P\Gamma U}(V)$. The case $\mathrm{PO}(V) \leq G \leq \mathrm{P\Gamma O}(V)$ is entirely similar. We know that $\mathrm{PU}(V) \lhd \mathrm{P\Gamma U}(V)$, and that $|\mathrm{P\Gamma U}(V) : \mathrm{PU}(V)|$ is finite, therefore $|G : \mathrm{PU}(V)|$ is also finite. Also, G is transitive on $\hat{P}$.

We claim that for G acting on $\mathrm{PG}(V)$, $\mathcal{O} \subseteq \mathrm{acl}(\hat{P})$. By 4.17, we know that for all $p \in \mathcal{O}$ there is $\bar{q} \in \bar{P}$ such that $\mathrm{PU}(V)_{\bar{q}}$ fixes p. We want to prove that p has finitely many translates under $G_{\bar{q}}$. This is equivalent to proving that $|G_{\bar{q}} : G_{\bar{q}p}| < \aleph_0$. Suppose for a contradiction that there are $(g_i : i \in \omega)$ which all lie in different cosets of $G_{\bar{q}p}$ in $G_{\bar{q}}$. Then the elements $g_i g_j^{-1}$ are all in $G_{\bar{q}}$ but not in $G_{\bar{q}p}$, hence they are not in $\mathrm{PU}(V)$. So we get that the $g_i, i \in \omega$ all lie in different cosets of $\mathrm{PU}(V)$ in G, which contradicts the fact that $|G : \mathrm{PU}(V)|$ is finite.

Next we show that $\mathcal{O} \subseteq \mathrm{dcl}(\hat{P})$. Suppose for a contradiction that $\mathcal{O}$ is not definable over $\hat{P}$. Then there is $g \in G$, $g \neq \mathrm{id}$ such that $g|_{\hat{P}} = \mathrm{id}$ (as in 4.17, by a König's lemma argument). It follows that $G_{\hat{P}}$ is nontrivial. Since $\hat{P}$ is an orbit, $G_{\hat{P}} \lhd G$. But $G_{\hat{P}} \leq G_p$ for any $p \in \hat{P}$. Since $|G : G_p| = |\mathrm{cos}(G : G_p)| = |\hat{P}| = \aleph_0$, $|G : G_{\hat{P}}|$ is infinite. But this is a contradiction, as G has no closed normal subgroups of infinite index. Indeed, if $H \lhd G$ is a closed nontrivial normal subgroup of infinite index, then $H \cap \mathrm{PU}(V)$ is a proper nontrivial closed normal subgroup of $\mathrm{PU}(V)$, a contradiction by [**9**]. Faithfulness of G follows as in 4.17. □

Corollary 4.27. *If G is a closed group acting on* $\mathrm{PG}(V)$ *such that* $\mathrm{PU}(V) \leq G \leq \mathrm{P\Gamma U}(V)$ *(resp.* $\mathrm{PO}(V) \leq G \leq \mathrm{P\Gamma O}(V)$*), then $\langle G, \mathrm{PG}(V)\rangle$ has a generalised weak $\forall\exists$ interpretation.*

Proof. By 4.18, 4.20, 4.21, 4.25, 4.17, there is a weak $\forall\exists$ interpretation for $\langle \mathrm{PU}(V), \hat{P}\rangle$ (resp. $\langle \mathrm{PO}(V), \hat{P}\rangle$). Since $\mathrm{PU}(V) \lhd \mathrm{P\Gamma U}(V)$ (resp. $\mathrm{PO}(V) \lhd \mathrm{P\Gamma O}(V)$), we can apply 2.2 to obtain a weak $\forall\exists$ interpretation for $\langle G, \hat{P}\rangle$. By 4.26, this yields a generalised weak $\forall\exists$ interpretation for $\langle G, \mathrm{PG}(V)\rangle$. □

5. *A reconstruction result for affine spaces*

In what follows we shall give an interpretability result for the general case of a primitive ω-categorical structure whose automorphism group has an abelian subgroup which is transitive on the structure. This result applies to the affine group $\mathrm{AGL}(V)$ of affine transformations of V, and it proves that V as an affine space is interpretable in $\mathrm{AGL}(V)$. We assume V to be ω-dimensional over a finite field F, as before.

Let us recall the basic definitions and notation about the affine group $\mathrm{AGL}(V)$. An affine transformation on V is a map $T_{M,b}$ of the form

$$vT_{M,b} := vM + b$$

where $M \in \mathrm{GL}(V)$ and $b \in V$. Then $\mathrm{AGL}(V)$ is the group of affine transformations on V. The affine group acts on V in the obvious way. Moreover, $\langle \mathrm{AGL}(V), V\rangle$ is an ω-categorical structure and the action of $\mathrm{AGL}(V)$ on V is primitive and faithful.

The affine transformations of the form $T_{I,b}$ where I is the identity in $\mathrm{GL}(V)$ are called the **translations** and they form a normal subgroup $\mathrm{T}(V) \lhd \mathrm{AGL}(V)$.

Also, the multiplicative group $\mathrm{T}(V)$ is isomorphic to the additive group V, so $\mathrm{T}(V)$ is abelian. By identifying $T_{M,0} \in \mathrm{AGL}(V)$ with $M \in \mathrm{GL}(V)$ and $T_{I,b} \in \mathrm{T}(V)$ with $b \in V$ it is easy to see that every element of $\mathrm{AGL}(V)$ can be expressed uniquely as the product of an element of $\mathrm{GL}(V)$ and an element of V. Moreover, $\mathrm{GL}(V) = \mathrm{Stab}_{\mathrm{AGL}(V)}(0)$, so $\mathrm{GL}(V) \leq \mathrm{AGL}(V)$, and we can write

$$\begin{aligned} \mathrm{AGL}(V) &= \mathrm{T}(V) \rtimes \mathrm{GL}(V) \\ &= V \rtimes \mathrm{GL}(V). \end{aligned}$$

We shall give our interpretability result in the general setting of an oligomorphic primitive permutation group G acting on a countable set X and having an abelian normal subgroup A. We shall show that then the structure on X is interpretable in G. This result applies to the affine group if we take $G = \mathrm{AGL}(V)$, $A = \mathrm{T}(V)$ and $X = V$. We list some folklore facts about the action of G on X. The proofs are straightforward, and some of them can be found in [**2**].

Lemma 5.1. *If G is faithful and primitive on X and $A \lhd G$ is non trivial, then A is transitive.*

Lemma 5.2. *Any transitive abelian permutation group H acting on a set X is regular.*

So if $A \lhd G$ is a non trivial normal subgroup, then A is transitive on X (by 5.1). If A is also abelian, then by 5.2 A is regular on X.

Proposition 5.3. *Let G be a primitive faithful group acting on a set X, and let A be a non trivial abelian normal subgroup. Let $\alpha \in X$ and let G_α be the stabilizer of α. Then $G = A \rtimes G_\alpha$.*

We now show that a suitable identification allows us to regard A as a copy of X in the group G.

Proposition 5.4. *Let X, G, A and α be as above. Then*

$$(G_\alpha, X) \cong (G_\alpha, A),$$

where G_α acts on A by conjugation.

Proof. Consider the map $\theta : A \to X$ defined by

$$\theta : 1 \to \alpha$$

$$\theta : g \to \alpha^g.$$

We claim θ defines an isomorphism between the natural action of G_α on X and the action of G_α on A by conjugation.

Now:

$$\begin{aligned} \alpha^g = \alpha^h &\Rightarrow \alpha^{gh^{-1}} = \alpha \\ &\Rightarrow gh^{-1} \in G_\alpha \cap A \\ &\Rightarrow gh^{-1} = 1 \quad \text{(since, by 5.3, } G_\alpha \cap A = \{1\}) \\ &\Rightarrow g = h \end{aligned}$$

so θ is injective. Since A is transitive, θ is also surjective.

Since $(a^g)\theta = \alpha^{g^{-1}ag} = \alpha^{ag}$ (as $g^{-1} \in G_\alpha$) $= [a\theta]^g$, θ is also a G_α-morphism, as required. $\square$

Proposition 5.5. *Let G, X, A and α be as above, and suppose further that G is oligomorphic on X. Then X with its structure is interpretable in G.*

Proof. We start by showing that the set X is definable in G. Note that since G is primitive, X has no non trivial proper blocks, i.e. no non trivial proper subset Y such that for all $g \in G$ either $Y^g = Y$ or $Y \cap Y^g = \emptyset$. Via the identification of X and A given in the proof of 5.4, this means that A has no non trivial proper subgroups that are G-invariant. So A is minimal among non trivial normal subgroups.

So choose $g \in A$, $g \neq 1$. We claim that $A = \{\prod_{i \in I}(g^{\varepsilon_i})^{h_i} : h_i \in G,\ \varepsilon_i = \pm 1\}$.

Let $H = \{\prod(g^{\varepsilon_i})^{h_i}\}$: clearly, $\{1\} \neq H \leq G$ and $H \subseteq A$. Now pick $h \in G$ and $\prod(g^{\varepsilon_i})^{h_i} \in H$. Then $(\prod(g^{\varepsilon_i})^{h_i})^h = \prod(g^{\varepsilon_i})^{h_i h} \in H$. So $H \lhd G$. By minimality of A among non trivial normal subgroups, $H = A$.

We now claim that there is a bound on the number of conjugates of g into which an element of A factors. We know that G is oligomorphic on X hence it is oligomorphic in its action on A as a pure set inherited from X via the identification of X and A, so in particular it has a finite number of orbits on A^2. Therefore the centraliser $C_G(g)$ has finitely many orbits on A. But elements which require a different number of products of conjugates lie in different orbits. So our claim follows, and A is definable.

Now consider the map $\phi : G \to G_\alpha$ given by $g\phi = (ah)\phi = h$, where $a \in A, h \in G_\alpha$ are the unique decomposition of g as an element of the semidirect product $A \rtimes G_\alpha$. Then ϕ is an epimorphism with kernel A, so that

$$G/A \cong G_\alpha.$$

We define an action of $A \rtimes G/A$ on A as follows:

$$a^{bAh} := (ab)^h \text{ for all } a, b \in A, Ah \in G/A.$$

For ease of notation, we shall write bh (rather than bAh) for the general element of $A \rtimes G/A$. Then:

$$(a^{bh})^{ck} = (h^{-1}abh)^{ck} = k^{-1}h^{-1}abhck$$

and

$$a^{bhck} = a^{bhch^{-1}hk} = k^{-1}h^{-1}abhch^{-1}hk = k^{-1}h^{-1}abhck$$

so that we have indeed defined an action. By using the isomorphism between G_α and G/A, we can identify the actions $\langle A \rtimes G/A, A\rangle$ and $\langle A \rtimes G_\alpha, A\rangle$. Then the structure on X is given by the orbits of $A \rtimes G/A$ on A^n for all $n \in \omega$. $\square$

Proposition 5.5 applies to many subgroups of $\mathrm{Sym}(\Omega)$. Indeed, it applies to all the primitive smoothly approximable structures of affine type described in [**10**].

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