Reconstruction of homogeneous relational structures

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The question: if we know the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a first order structure $\mathcal{M}$, what do we know about $\mathcal{M}$?

The question is only sensible in highly symmetric contexts, such as ω-**categorical structures**.

If $\mathcal{M}$ and $\mathcal{N}$ ω-categorical:

1. $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$ as topological groups $\iff$ $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable;

2. $\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M} \rangle \cong \langle \mathrm{Aut}(\mathcal{N}), \mathcal{N} \rangle \iff \mathcal{M}$ and $\mathcal{N}$ have the same 0-definable sets.

Reconstruction results give conditions under which the pure group structure determines the topology (e.g. the small index property) or the action of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{M}$.

Rubin's reconstruction method

Definition 1 (Weak $\forall\exists$ interpretation) *Let $\mathcal{M}$ be ω-categorical and transitive. We look for:*

1. *a conjugacy class $C \subseteq \mathrm{Aut}(\mathcal{M})$, say $C = g^{\mathrm{Aut}(\mathcal{M})}$;*

2. *an equivalence relation E on C which is*

   - $\forall\exists$ *definable* **in the language of groups**, *possibly with parameters, and*

   - *invariant under conjugacy by an element of* Aut($\mathcal{M}$), *that is: for all $h, k \in C$, $g \in$* Aut($\mathcal{M}$)

   $$hEk \iff h^g E k^g;$$

3. *a bijection $\phi : \mathcal{M} \to C/E$ such that*

   $$\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M} \rangle \cong \langle \mathrm{Aut}(\mathcal{M}), C/E \rangle,$$

   where Aut($\mathcal{M}$) *acts on C/E by conjugation.*

If $\langle C, E, \phi \rangle$ can be found, we say that $\mathcal{M}$ has a **weak $\forall\exists$ interpretation**.

Generally, C contains maps having a single fixed point, and E is "has the same fixed point as".

When $\mathcal{M}$ is not transitive, a weak $\forall\exists$ interpretation for $\mathcal{M}$ consists of weak $\forall\exists$ interpretations for each orbit of Aut($\mathcal{M}$) on $\mathcal{M}$.

Theorem 2 (M. Rubin, 92) *Let $\mathcal{M}$ and $\mathcal{N}$ be ω-categorical structures without algebraicity, and suppose that $\mathcal{M}$ has a weak $\forall\exists$ interpretation. Suppose that*

$$\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$$

as pure groups. Then

$$\langle \mathrm{Aut}(\mathcal{M}), \mathcal{M} \rangle \cong \langle \mathrm{Aut}(\mathcal{N}), \mathcal{N} \rangle,$$

i.e. $\mathcal{M}$ and $\mathcal{N}$ have the same 0-definable sets.

A weak $\forall\exists$ interpretation for certain homogeneous relational ω-categorical structures

Let $\mathcal{M}$ be an L-structure, where L only contains relation symbols. Suppose that $\mathcal{M}$ is transitive, ω-categorical and homogeneous. Let $G = \mathrm{Aut}(\mathcal{M})$ and $d \in \mathcal{M}$. Let

$$X_d := \{g \in G : \mathrm{fix}(g) = d\}$$

$$G_d := \{g \in G : d^g = d\}$$

Then X_d and $X_d \times X_d$ are Polish spaces. Moreover $X_d^{G_d} = X_d$.

Definition 3 *A pair (f_1, f_2) is G_d-**generic** if the orbit $(f_1, f_2)^{G_d}$ is comeagre in $X_d \times X_d$.*

Fact 4 *Let (f_1, f_2) be G_d-generic in $X_d \times X_d$. Let $\mathcal{D} := (f_1, f_2)^{G_d}$. Then:*

1. $f_1^{G_d} = f_2^{G_d}$ is comeagre in X_d;
2. for all $g \in f_1^{G_d}$, the fibre

$$\mathcal{D}_g = \{f \in X_d : (g, f) \in \mathcal{D}\}$$

is comeagre in X_d.

Let $\mathcal{D}^G := \{(f_1, f_2)^g : g \in G\}$. Then $\mathcal{D}^G$ consists of pairs (h_1, h_2) such that $\text{fix}(h_1) = \text{fix}(h_2)$ is a singleton.

Proposition 5 *Let E be the following equivalence relation on f_1^G:*

$$h_1 E h_2 \iff \text{fix}(h_1) = \text{fix}(h_2).$$

Then E is $\exists$-definable in G by

$$h_1 E h_2 \iff \exists f \in G : (h_1, f),\ (h_2, f) \in \mathcal{D}^G.$$

Proof $(\Leftarrow)$ trivial.

$(\Rightarrow)$ If $\text{fix}(h_1) = \text{fix}(h_2) = e$, find $k \in G$ such that $e^k = d$. Then $\text{fix}(h_1^k) = \text{fix}(h_2^k) = d$, so $h_1, h_2 \in f_1^{G_d}$. Then the fibres $\mathcal{D}_{h_1^k}, \mathcal{D}_{h_2^k}$ are both comeagre. Hence there is $\widehat{f} \in \mathcal{D}_{h_1^k} \cap \mathcal{D}_{h_2^k}$. Let $f := \widehat{f}^{k^{-1}}$.

It is easy to check that E is $\exists$ definable. $\square$

Conditions for the existence of a G_d-generic pair

Let $\mathcal{M}$ be an ω-categorical, transitive and homogeneous structure in the relational language L, let $d \in \mathcal{M}$.

Let κ be the class of finite substructures $\mathcal{A}$ of $\mathcal{M}$ such that there are $f_1, f_2 \in \mathrm{Aut}(\mathcal{A})$ with $\mathrm{fix}(f_1) = \mathrm{fix}(f_2) = d$.

Suppose κ has the *free amalgamation property*.

Suppose further the following holds (**fixed point extension property**):
for all finite $\mathcal{A} \subseteq \mathcal{M}$, if $p_1, \ldots, p_n$ are finite partial isomorphisms of $\mathcal{A}$ and $\mathrm{fix}(p_1) = \mathrm{fix}(p_2) = \cdots = \mathrm{fix}(p_n) = \{d\}$, there are a finite $\mathcal{B} \subseteq \mathcal{M}$ and $f_1, \ldots, f_n \in \mathrm{Aut}(\mathcal{B})$ such that:

- $p_i \subseteq f_i$;
- $\mathrm{fix}(f_i) = \{d\}$ for $i = 1, \ldots, n$.

Then there is a countable homogeneous structure

$$(\mathcal{M}, f_1, f_2, d)$$

which embeds every member of κ (the *Fraïssé limit* of κ).

The pair (f_1, f_2) is G_d-generic.

(NB the formal construction requires a change of language. The L-structure on $(\mathcal{M}, f_1, f_2, d)$ is the $\mathcal{M}$ we started off with).

Theorem 6 *Let $\mathcal{M}$ be homogeneous, ω-categorical and transitive in a finite relational language. Let κ, f_1, f_2, d be as above (so κ has the free amalgamation property and the fixed point extension property). Let $(\mathcal{M}, f_1, f_2, d)$ be the Fraïssé limit of κ, and let $G := \mathrm{Aut}(\mathcal{M})$, $\mathcal{D} := (f_1, f_2)^{G_d}$. Then $\mathcal{D}$ is comeagre in $X_d \times X_d$, i.e. (f_1, f_2) is a G_d-generic pair.*

Proof (sketch) We play the Banach-Mazur game of $\mathcal{D}$: players I and II choose an increasing sequence

$$(p_{1,0},\ p_{2,0}) \subseteq (p_{1,1},\ p_{2,1}) \subseteq (p_{1,2},\ p_{2,2}), \ldots$$

of finite partial isomorphisms of $\mathcal{M}$ such that $\mathrm{fix}(p_{ij}) = \{d\}$ for all i, j.

Player I starts the game. Player II wins if and only if $(p_1, p_2) := (\bigcup_{i\in\omega} p_{1,i}, \bigcup_{i\in\omega} p_{2,i}) \in \mathcal{D}$. Player II has a winning strategy iff $\mathcal{D}$ is comeagre in $X_d \times X_d$.

Player II can always play so that at stage i, for $i > 1$ and even,

1. he can choose to put any $x \in \mathcal{M}$ in the domain and range of $p_{1,i}, p_{2,i}$;

2. $(p_{1,i}, p_{2,i}) \in P^2$ and $\mathrm{dom}(p_{1,i}) = \mathrm{ran}(p_{1,i})$, $\mathrm{dom}(p_{2,i}) = \mathrm{ran}(p_{2,i})$;

3. he eventually gets that $(\mathcal{M}, p_1, p_2, d)$ is homogeneous, so
$$(p_1, p_2) \sim (f_1, f_2).$$

The fixed point extension property (FEP) is crucial for all of 1., 2. and 3.

At stage $i+1$, i even, player II is given a finite structure $(\Delta_i, p_{1,i}, p_{2,i}, d)$, where the $p_{j,i}$ are finite partial isomorphisms of Δ_i. For points 1. and 2., for any $x \in \mathcal{M}$, II can consider $\Delta'_{i+1} := \Delta_i \cup \{x\}$ and use FEP to obtain extensions Δ_{i+1} of Δ'_{i+1}, and $p_{1,i+1}, p_{2,i+1} \in \mathrm{Aut}(\Delta_{i+1})$ of $p_{1,i}, p_{2,i}$, each fixing only d.

Some structures covered by this method (i.e. for which the fixed point extension property holds):

1. the class of all finite structures in a given finite relational language S (e.g. k-hypergraphs);
2. K_m-free graphs;
3. Henson digraphs;
4. the arity k analogues of triangle-free graphs (k-hypergraphs not admitting a $k+1$ set all of whose k-tuples are hyperedges.)

Extension properties for finite partial isomorphisms — with no restrictions on the cycle type — have been proved for graphs first (Hrushovski), and subsequently for the other structures mentioned above (Herwig).

They yield the small index property for the structures concerned.

The proofs of FEP are essentially the same.