Model theory of Steiner triple systems

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UMI-SIMAI PTM Joint Meeting, September 2018

Steiner triple systems

Definition

A finite **Steiner triple system** *(STS) of order n is a pair $(V, \mathcal{B})$ where:*

- *V is a set of n elements;*
- *$\mathcal{B}$ is a collection of 3-element subsets of V (the* **blocks***) such that any two $x, y \in V$ are contained in exactly one block.*

A set V with a collection of 3-element subsets is a **partial STS** *if any two elements of V belong to at most one block.*

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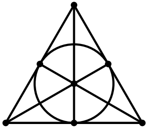

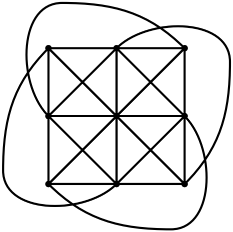

Kirkman's schoolgirl problem

Fifteen girls in a school take a walk in rows of three for seven days in succession. Is there an arrangement such that no two girls walk together in a row more than once?
(Thomas Penyngton Kirkman, 1850)

STSs appear in

- combinatorial design theory (they are balanced incomplete block designs)
- design of experiments
- coding theory.

More general Steiner systems are connected to the Mathieu groups.

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More general Steiner systems are connected to the Mathieu groups.

- When n is finite, an STS of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.
- If we allow $|V| \geq \omega$, the pair $(V, \mathcal{B})$ is an **infinite STS**.

We can describe blocks via

- a ternary relation R where $R(x, y, z)$ if and only if $\{x, y, z\}$ is a block, or
- a binary operation $\cdot$ defined by

$$x \cdot y = z \text{ iff } \{x, y, z\} \text{ is a block.}$$

When blocks are described by a relation, a substructure of an STS is a *partial* STS.
In a functional language, substructures are STSs.

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In a functional language, substructures are STSs.

STS axioms

We choose a functional language, so that an STS is a structure $(A, \cdot)$ where $\cdot$ is a binary operation on A such that

1. $x \cdot y = y \cdot x$
2. $x \cdot x = x$
3. $x \cdot (x \cdot y) = y$.

Definition

T_{STS} is the theory that contains axioms 1–3 above.

T_{STS} is a universal theory.

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Extension properties

Fact

1. *Every finite partial STS can be embedded in a finite STS.*
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The Fraïssé limit of the finite Steiner triple systems

The class $\mathcal{C}$ of all finite Steiner triple systems has

- the Joint Embedding and the Amalgamation Properties
- the Hereditary Property
- countably many isomorphism types.

Therefore $\mathcal{C}$ has a Fraïssé limit: the unique (up to isomorphism) countable Steiner triple system M_F which is *ultrahomogeneous* and *universal* (for finite Steiner triple systems).

M_F is locally finite. It is not ω-categorical.

Questions

What can we say about $\mathrm{Th}(M_F)$*? Can we describe its models? Does it have q.e.?*

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Axiomatising $\mathrm{Th}(M_F)$

Definition

Let B be a finite partial STS. Then

- δ_B *is a formula that describes the diagram of* B
- $A \subseteq B$ *is* **relatively closed** *in B if for every $a, b \in A$ and $c \in B$, if $a \cdot b = c$ then $c \in A$.*

Definition

If B is a finite partial STS and $A \subseteq B$ a relatively closed subset, then

$$\phi_{(A,B)} = \forall \bar{x} \left(\delta_A(\bar{x}) \rightarrow \exists \bar{y}\, \delta_B(\bar{x}, \bar{y}) \right).$$

Let $\Delta = \{\phi_{(A,B)} : B$ is a finite partial STS and $A \subseteq B$ is a relatively closed subset$\}$.

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The theory of M_F

Let $T^*_{\mathrm{STS}} = \Delta \cup T_{\mathrm{STS}}$.

Fact

$M_F \models T^*_{\mathrm{STS}}$.

There is more.

Theorem

The theory T^*_{STS}

- *axiomatises the existentially closed Steiner triple systems*
- *is model complete*
- *is the model companion of T_{STS}*
- *is complete*
- *has quantifier elimination.*

M_F is a prime model of T^*_{STS}.

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Proof (sketch).

- T_{STS} is universal, so every model extends to an e.c. model
- T^*_{STS} axiomatises the e.c. models of T_{STS}

Therefore T^*_{STS} is the model companion of T_{STS}.

In particular, T^*_{STS} is model complete.

T^*_{STS} has the joint embedding property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} is complete.

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More on T^*_{STS}

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The theory T^*_{STS}

- *is not small*
- *is* TP_2
- *is* NSOP_1
- *has elimination of hyperimaginaries and weak elimination of imaginaries.*

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TP_2

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A formula $\varphi(\bar{x};\bar{y})$ *has the tree property of the second kind (* TP_2*) in* T *if in the monster model of* T *there is an array of tuples* $(\bar{a}_{ij} \mid i,j < \omega)$ *and some natural number* k *such that*

- *for each* $i < \omega$ *the set* $\{\varphi(\bar{x},\bar{a}_{ij}) \mid j < \omega\}$ *is* k*-inconsistent*
- *for each* $f : \omega \to \omega$ *the path* $\{\varphi(\bar{x},\bar{a}_{if(i)}) \mid i < \omega\}$ *is consistent.*

We say that T *is* TP_2 *if some formula has* TP_2 *in* T*.*

$$\begin{array}{ccccc}
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Proposition

The formula

$$\varphi(x; y_1, y_2, y_3) \equiv \quad x = (y_1 \cdot (y_2 \cdot (y_3 \cdot x)))$$

has TP_2 *in* T^*_{Sq}.

Proof (sketch).

We build

- an array $(a_i b_i c_{ij} \mid i, j < \omega)$
- a sequence $(d_f \mid f \in \omega^\omega)$

and define a partial STS such that

- for each $i \in \omega$, the set $\{\varphi(x; a_i, b_i, c_{ij}) \mid j < \omega\}$ is 2-inconsistent
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- for each $i \in \omega$, the set $\{\varphi(x; a_i, b_i, c_{ij}) \mid j < \omega\}$ is 2-inconsistent
- for each $f \in \omega^\omega$, the element d_f realizes $\{\varphi(x; a_i, b_i, c_{if(i)}) \mid i < \omega\}$.

Proposition

The formula

$$\varphi(x; y_1, y_2, y_3) \equiv \quad x = (y_1 \cdot (y_2 \cdot (y_3 \cdot x)))$$

has TP_2 *in* T^*_{Sq}.

Proof (sketch).

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We use the following cancellation law:

$$\forall xyz\,(x \cdot y = x \cdot z \rightarrow y = z)\,.$$

We choose an array

$$\begin{array}{cccc} a_0 b_0 c_{00} & a_0 b_0 c_{01} & a_0 b_0 c_{02} & \dots \\ a_1 b_1 c_{10} & a_1 b_1 c_{11} & a_1 b_1 c_{12} & \dots \\ \vdots & \vdots & \vdots & \end{array}$$

where the entries are pairwise distinct. Then for $j \neq k$ the formula

$$\varphi(x, a_i, b_i, c_{ij}) \wedge \varphi(x, a_i, b_i, c_{ik})\,.$$

is inconsistent, as

$$a_i \cdot (b_i \cdot (c_{ij} \cdot x)) = a_i \cdot (b_i \cdot (c_{ik} \cdot x))$$

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But given $f \in \omega^\omega$, we can choose d_f and construct a partial STS such that, for all $i \in \omega$

$$d_f = a_i \cdot (b_i \cdot (c_{i\,f(i)} \cdot x)) .$$

This is achieved as follows:

- for i, j such that $f(i) = j$, add points a^*_{ijf} and b^*_{ijf}
- define the product on $\{d_f, a_i, b_i, c_{ij}, a^*_{ijf}, b^*_{ijf}\}$ so that

$$d_f = a_i \cdot a^*_{ijf} = a_i \cdot (b_i \cdot b^*_{ijf}) = a_i \cdot (b_i \cdot (c_{ij} \cdot d_f)) ,$$

As i ranges over ω and f over ω^ω, we obtain a partial STS. This embeds in the monster model. □