The theory of Steiner triple systems

Silvia Barbina

joint work with Enrique Casanovas

LC2018, Udine

Steiner triple systems

Definition

A finite **Steiner triple system** *(STS) of order n is a pair $(V, \mathcal{B})$ where:*

- *V is a set of n elements;*
- *$\mathcal{B}$ is a collection of 3-element subsets of V (the* **blocks***) such that any two $x, y \in V$ are contained in exactly one block.*

A set V with a collection of 3-element subsets is a **partial STS** *if any two elements of V belong to at most one block.*

Steiner triple systems

Definition

A finite **Steiner triple system** *(STS) of order n is a pair* $(V, \mathcal{B})$ *where:*

- V *is a set of n elements;*
- $\mathcal{B}$ *is a collection of 3-element subsets of* V *(the* **blocks***) such that any two* $x, y \in V$ *are contained in exactly one block.*

A set V *with a collection of 3-element subsets is a* **partial STS** *if any two elements of* V *belong to at most one block.*

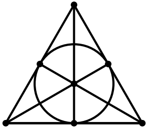
Steiner triple systems

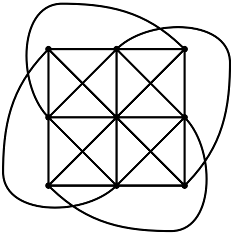
Definition

A finite **Steiner triple system** *(STS) of order n is a pair $(V, \mathcal{B})$ where:*

- *V is a set of n elements;*
- *$\mathcal{B}$ is a collection of 3-element subsets of V (the* **blocks***) such that any two $x, y \in V$ are contained in exactly one block.*

A set V with a collection of 3-element subsets is a **partial STS** *if any two elements of V belong to at most one block.*





Steiner triple systems

Kirkman's schoolgirl problem

Fifteen girls in a school take a walk in rows of three for seven days in succession. Is there an arrangement such that no two girls walk together in a row more than once?
(Thomas Penyngton Kirkman, 1850)

STSs appear in

- combinatorial design theory (they are balanced incomplete block designs)
- design of experiments
- coding theory.

More general Steiner systems are connected to the Mathieu groups.

Steiner triple systems

Kirkman's schoolgirl problem

Fifteen girls in a school take a walk in rows of three for seven days in succession. Is there an arrangement such that no two girls walk together in a row more than once?
(Thomas Penyngton Kirkman, 1850)

STSs appear in

- combinatorial design theory (they are balanced incomplete block designs)
- design of experiments
- coding theory.

More general Steiner systems are connected to the Mathieu groups.

Steiner triple systems

Kirkman's schoolgirl problem

Fifteen girls in a school take a walk in rows of three for seven days in succession. Is there an arrangement such that no two girls walk together in a row more than once?
(Thomas Penyngton Kirkman, 1850)

STSs appear in

- combinatorial design theory (they are balanced incomplete block designs)
- design of experiments
- coding theory.

More general Steiner systems are connected to the Mathieu groups.

Steiner triple systems

Kirkman's schoolgirl problem

Fifteen girls in a school take a walk in rows of three for seven days in succession. Is there an arrangement such that no two girls walk together in a row more than once?
(Thomas Penyngton Kirkman, 1850)

STSs appear in

- combinatorial design theory (they are balanced incomplete block designs)
- design of experiments
- coding theory.

More general Steiner systems are connected to the Mathieu groups.

Steiner triple systems

Kirkman's schoolgirl problem

Fifteen girls in a school take a walk in rows of three for seven days in succession. Is there an arrangement such that no two girls walk together in a row more than once?
(Thomas Penyngton Kirkman, 1850)

STSs appear in

- combinatorial design theory (they are balanced incomplete block designs)
- design of experiments
- coding theory.

More general Steiner systems are connected to the Mathieu groups.

Steiner triple systems

- When n is finite, an STS of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.
- If we allow $|V| \geq \omega$, the pair $(V, \mathcal{B})$ is an **infinite STS**.

We can describe blocks via

- a ternary relation R where $R(x, y, z)$ if and only if $\{x, y, z\}$ is a block, or
- a binary operation $\cdot$ defined by

$$x \cdot y = z \text{ iff } \{x, y, z\} \text{ is a block.}$$

When blocks are described by a relation, a substructure of an STS is a *partial* STS.
In a functional language, substructures are STSs.

Steiner triple systems

- When n is finite, an STS of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.
- If we allow $|V| \geq \omega$, the pair $(V, \mathcal{B})$ is an **infinite STS**.

We can describe blocks via

- a ternary relation R where $R(x, y, z)$ if and only if $\{x, y, z\}$ is a block, or
- a binary operation $\cdot$ defined by

$$x \cdot y = z \text{ iff } \{x, y, z\} \text{ is a block.}$$

When blocks are described by a relation, a substructure of an STS is a *partial* STS.
In a functional language, substructures are STSs.

Steiner triple systems

- When n is finite, an STS of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.
- If we allow $|V| \geq \omega$, the pair $(V, \mathcal{B})$ is an **infinite STS**.

We can describe blocks via

- a ternary relation R where $R(x, y, z)$ if and only if $\{x, y, z\}$ is a block, or
- a binary operation $\cdot$ defined by

$$x \cdot y = z \text{ iff } \{x, y, z\} \text{ is a block.}$$

When blocks are described by a relation, a substructure of an STS is a *partial* STS.
In a functional language, substructures are STSs.

Steiner triple systems

- When n is finite, an STS of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.
- If we allow $|V| \geq \omega$, the pair $(V, \mathcal{B})$ is an **infinite STS**.

We can describe blocks via

- a ternary relation R where $R(x, y, z)$ if and only if $\{x, y, z\}$ is a block, or
- a binary operation $\cdot$ defined by

$$x \cdot y = z \text{ iff } \{x, y, z\} \text{ is a block.}$$

When blocks are described by a relation, a substructure of an STS is a *partial* STS.
In a functional language, substructures are STSs.

Steiner triple systems

- When n is finite, an STS of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.
- If we allow $|V| \geq \omega$, the pair $(V, \mathcal{B})$ is an **infinite STS**.

We can describe blocks via

- a ternary relation R where $R(x, y, z)$ if and only if $\{x, y, z\}$ is a block, or
- a binary operation $\cdot$ defined by

$$x \cdot y = z \text{ iff } \{x, y, z\} \text{ is a block.}$$

When blocks are described by a relation, a substructure of an STS is a *partial* STS.
In a functional language, substructures are STSs.

Steiner triple systems

- When n is finite, an STS of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.
- If we allow $|V| \geq \omega$, the pair $(V, \mathcal{B})$ is an **infinite STS**.

We can describe blocks via

- a ternary relation R where $R(x, y, z)$ if and only if $\{x, y, z\}$ is a block, or
- a binary operation $\cdot$ defined by

$$x \cdot y = z \text{ iff } \{x, y, z\} \text{ is a block.}$$

When blocks are described by a relation, a substructure of an STS is a *partial* STS.
In a functional language, substructures are STSs.

Steiner triple systems

- When n is finite, an STS of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.
- If we allow $|V| \geq \omega$, the pair $(V, \mathcal{B})$ is an **infinite STS**.

We can describe blocks via

- a ternary relation R where $R(x, y, z)$ if and only if $\{x, y, z\}$ is a block, or
- a binary operation $\cdot$ defined by

$$x \cdot y = z \text{ iff } \{x, y, z\} \text{ is a block.}$$

When blocks are described by a relation, a substructure of an STS is a *partial* STS.
In a functional language, substructures are STSs.

STS axioms

We choose a functional language, so that an STS is a structure $(A, \cdot)$ where $\cdot$ is a binary operation on A such that

1. $x \cdot y = y \cdot x$
2. $x \cdot x = x$
3. $x \cdot (x \cdot y) = y$.

Definition

T_{STS} is the theory that contains axioms 1–3 above.

T_{STS} is a universal theory.

STS axioms

We choose a functional language, so that an STS is a structure $(A, \cdot)$ where $\cdot$ is a binary operation on A such that

1. $x \cdot y = y \cdot x$
2. $x \cdot x = x$
3. $x \cdot (x \cdot y) = y$.

Definition

T_{STS} is the theory that contains axioms 1–3 above.

T_{STS} is a universal theory.

STS axioms

We choose a functional language, so that an STS is a structure $(A, \cdot)$ where $\cdot$ is a binary operation on A such that

1. $x \cdot y = y \cdot x$
2. $x \cdot x = x$
3. $x \cdot (x \cdot y) = y$.

Definition

T_{STS} is the theory that contains axioms 1–3 above.

T_{STS} is a universal theory.

Extension properties

Fact

1. *Every finite partial STS can be embedded in a finite STS.*
2. *Every infinite partial STS can be embedded in an STS of the same cardinality.*

Extension properties

Fact

1. *Every finite partial STS can be embedded in a finite STS.*
2. *Every infinite partial STS can be embedded in an STS of the same cardinality.*

The Fraïssé limit of the finite Steiner triple systems

The class $\mathcal{C}$ of all finite Steiner triple systems has

- the Joint Embedding and the Amalgamation Properties
- the Hereditary Property
- countably many isomorphism types.

Therefore $\mathcal{C}$ has a Fraïssé limit: the unique (up to isomorphism) countable Steiner triple system M_F which is *ultrahomogeneous* and *universal* (for finite Steiner triple systems).

Questions

What can we say about $\mathrm{Th}(M_F)$*? Can we describe its models? Does it have q.e.?*

The Fraïssé limit of the finite Steiner triple systems

The class $\mathcal{C}$ of all finite Steiner triple systems has

- the Joint Embedding and the Amalgamation Properties
- the Hereditary Property
- countably many isomorphism types.

Therefore $\mathcal{C}$ has a Fraïssé limit: the unique (up to isomorphism) countable Steiner triple system M_F which is *ultrahomogeneous* and *universal* (for finite Steiner triple systems).

Questions

What can we say about $\mathrm{Th}(M_F)$*? Can we describe its models? Does it have q.e.?*

The Fraïssé limit of the finite Steiner triple systems

The class $\mathcal{C}$ of all finite Steiner triple systems has

- the Joint Embedding and the Amalgamation Properties
- the Hereditary Property
- countably many isomorphism types.

Therefore $\mathcal{C}$ has a Fraïssé limit: the unique (up to isomorphism) countable Steiner triple system M_F which is *ultrahomogeneous* and *universal* (for finite Steiner triple systems).

Questions

What can we say about $\mathrm{Th}(M_F)$*? Can we describe its models? Does it have q.e.?*

The Fraïssé limit of the finite Steiner triple systems

The class $\mathcal{C}$ of all finite Steiner triple systems has

- the Joint Embedding and the Amalgamation Properties
- the Hereditary Property
- countably many isomorphism types.

Therefore $\mathcal{C}$ has a Fraïssé limit: the unique (up to isomorphism) countable Steiner triple system M_F which is *ultrahomogeneous* and *universal* (for finite Steiner triple systems).

Questions

What can we say about $\mathrm{Th}(M_F)$*? Can we describe its models? Does it have q.e.?*

The Fraïssé limit of the finite Steiner triple systems

The class $\mathcal{C}$ of all finite Steiner triple systems has

- the Joint Embedding and the Amalgamation Properties
- the Hereditary Property
- countably many isomorphism types.

Therefore $\mathcal{C}$ has a Fraïssé limit: the unique (up to isomorphism) countable Steiner triple system M_F which is *ultrahomogeneous* and *universal* (for finite Steiner triple systems).

Questions

What can we say about $\mathrm{Th}(M_F)$*? Can we describe its models? Does it have q.e.?*

Axiomatising Th(M_F)

Definition

Let B be a finite partial STS. Then

- *δ_B is a formula that describes the diagram of B*
- *$A \subseteq B$ is* **relatively closed** *in B if for every $a, b \in A$ and $c \in B$, if $a \cdot b = c$ then $c \in A$.*

Definition

If B is a finite partial STS and $A \subseteq B$ a relatively closed subset, then

$$\phi_{(A,B)} = \forall \bar{x} \left(\delta_A(\bar{x}) \to \exists \bar{y}\, \delta_B(\bar{x}, \bar{y}) \right).$$

Let $\Delta = \{\phi_{(A,B)} : B$ is a finite partial STS and $A \subseteq B$ is a relatively closed subset$\}$.

Axiomatising Th(M_F)

Definition

Let B be a finite partial STS. Then

- *δ_B is a formula that describes the diagram of B*
- *$A \subseteq B$ is* **relatively closed** *in B if for every $a, b \in A$ and $c \in B$, if $a \cdot b = c$ then $c \in A$.*

Definition

If B is a finite partial STS and $A \subseteq B$ a relatively closed subset, then

$$\phi_{(A,B)} = \forall \bar{x} \left(\delta_A(\bar{x}) \to \exists \bar{y}\, \delta_B(\bar{x}, \bar{y}) \right).$$

Let $\Delta = \{\phi_{(A,B)} : B$ is a finite partial STS and $A \subseteq B$ is a relatively closed subset$\}$.

Axiomatising Th(M_F)

Definition

Let B be a finite partial STS. Then

- δ_B *is a formula that describes the diagram of B*
- $A \subseteq B$ *is* **relatively closed** *in B if for every $a, b \in A$ and $c \in B$, if $a \cdot b = c$ then $c \in A$.*

Definition

If B is a finite partial STS and $A \subseteq B$ a relatively closed subset, then

$$\phi_{(A,B)} = \forall \bar{x} \left(\delta_A(\bar{x}) \rightarrow \exists \bar{y} \, \delta_B(\bar{x}, \bar{y}) \right).$$

Let $\Delta = \{\phi_{(A,B)} : B$ is a finite partial STS and $A \subseteq B$ is a relatively closed subset$\}$.

Axiomatising Th(M_F)

Definition

Let B be a finite partial STS. Then

- δ_B *is a formula that describes the diagram of* B
- $A \subseteq B$ *is* **relatively closed** *in* B *if for every* $a, b \in A$ *and* $c \in B$, *if* $a \cdot b = c$ *then* $c \in A$.

Definition

If B is a finite partial STS and $A \subseteq B$ a relatively closed subset, then

$$\phi_{(A,B)} = \forall \bar{x} \left(\delta_A(\bar{x}) \rightarrow \exists \bar{y} \, \delta_B(\bar{x}, \bar{y}) \right).$$

Let $\Delta = \{\phi_{(A,B)} : B$ *is a finite partial STS and* $A \subseteq B$ *is a relatively closed subset*$\}$.

Let $T^*_{\mathrm{STS}} = \Delta \cup T_{\mathrm{STS}}$.

Fact

$M_F \models T^*_{\mathrm{STS}}$.

There is more.

Theorem

The theory T^*_{STS}

- *axiomatises the existentially closed Steiner triple systems*
- *it is model complete*
- *is the model companion of T_{STS}*
- *is complete*
- *has quantifier elimination.*

M_F is a prime model of T^*_{STS}.

Let $T^*_{\mathrm{STS}} = \Delta \cup T_{\mathrm{STS}}$.

Fact

$M_F \models T^*_{\mathrm{STS}}$.

There is more.

Theorem

The theory T^*_{STS}

- *axiomatises the existentially closed Steiner triple systems*
- *it is model complete*
- *is the model companion of T_{STS}*
- *is complete*
- *has quantifier elimination.*

M_F is a prime model of T^*_{STS}.

Let $T^*_{\mathrm{STS}} = \Delta \cup T_{\mathrm{STS}}$.

Fact

$M_F \models T^*_{\mathrm{STS}}$.

There is more.

Theorem

The theory T^*_{STS}

- *axiomatises the existentially closed Steiner triple systems*
- *it is model complete*
- *is the model companion of T_{STS}*
- *is complete*
- *has quantifier elimination.*

M_F is a prime model of T^*_{STS}.

Let $T^*_{\mathrm{STS}} = \Delta \cup T_{\mathrm{STS}}$.

Fact

$M_F \models T^*_{\mathrm{STS}}$.

There is more.

Theorem

The theory T^*_{STS}

- *axiomatises the existentially closed Steiner triple systems*
- *it is model complete*
- *is the model companion of T_{STS}*
- *is complete*
- *has quantifier elimination.*

M_F is a prime model of T^*_{STS}.

Let $T^*_{\mathrm{STS}} = \Delta \cup T_{\mathrm{STS}}$.

Fact

$M_F \models T^*_{\mathrm{STS}}$.

There is more.

Theorem

The theory T^*_{STS}

- *axiomatises the existentially closed Steiner triple systems*
- *it is model complete*
- *is the model companion of* T_{STS}
- *is complete*
- *has quantifier elimination.*

M_F *is a prime model of* T^*_{STS}.

Let $T^*_{\mathrm{STS}} = \Delta \cup T_{\mathrm{STS}}$.

Fact

$M_F \models T^*_{\mathrm{STS}}$.

There is more.

Theorem

The theory T^*_{STS}

- *axiomatises the existentially closed Steiner triple systems*
- *it is model complete*
- *is the model companion of* T_{STS}
- *is complete*
- *has quantifier elimination.*

M_F *is a prime model of* T^*_{STS}.

Let $T^*_{\mathrm{STS}} = \Delta \cup T_{\mathrm{STS}}$.

Fact

$M_F \models T^*_{\mathrm{STS}}$.

There is more.

Theorem

The theory T^*_{STS}

- *axiomatises the existentially closed Steiner triple systems*
- *it is model complete*
- *is the model companion of T_{STS}*
- *is complete*
- *has quantifier elimination.*

M_F is a prime model of T^*_{STS}.

- T_{STS} is universal, so every model extends to an e.c. model
- T^*_{STS} axiomatises the e.c. models of T_{STS}

Therefore T^*_{STS} is the model companion of T_{STS}.

In particular, T^*_{STS} is model complete.

T^*_{STS} has the joint embedding property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} is complete.

T^*_{STS} has the amalgamation property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} has quantifier elimination.

- T_{STS} is universal, so every model extends to an e.c. model
- T^*_{STS} axiomatises the e.c. models of T_{STS}

Therefore T^*_{STS} is the model companion of T_{STS}.

In particular, T^*_{STS} is model complete.

T^*_{STS} has the joint embedding property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} is complete.

T^*_{STS} has the amalgamation property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} has quantifier elimination.

- T_{STS} is universal, so every model extends to an e.c. model
- T^*_{STS} axiomatises the e.c. models of T_{STS}

Therefore T^*_{STS} is the model companion of T_{STS}.

In particular, T^*_{STS} is model complete.

T^*_{STS} has the joint embedding property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} is complete.

T^*_{STS} has the amalgamation property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} has quantifier elimination.

- T_{STS} is universal, so every model extends to an e.c. model
- T^*_{STS} axiomatises the e.c. models of T_{STS}

Therefore T^*_{STS} is the model companion of T_{STS}.

In particular, T^*_{STS} is model complete.

T^*_{STS} has the joint embedding property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} is complete.

T^*_{STS} has the amalgamation property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} has quantifier elimination.

- T_{STS} is universal, so every model extends to an e.c. model
- T^*_{STS} axiomatises the e.c. models of T_{STS}

Therefore T^*_{STS} is the model companion of T_{STS}.

In particular, T^*_{STS} is model complete.

T^*_{STS} has the joint embedding property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} is complete.

T^*_{STS} has the amalgamation property (because T_{STS} has), and it is model complete. Therefore T^*_{STS} has quantifier elimination.

M_F and saturation

The Fraïssé limit M_F is not saturated: it does not realise the type of a finitely generated infinite STS.

Question

Does T^*_{STS} have a countable saturated model?

M_F and saturation

The Fraïssé limit M_F is not saturated: it does not realise the type of a finitely generated infinite STS.

Question

Does T^*_{STS} have a countable saturated model?

A free construction

Given three points $\{a, b, c\}$ not forming a block, define a chain $\{A_i : i < \omega\}$ of partial STSs inductively:

- $A_0 = \{a, b, c\}$ and no products are defined
- $A_{i+1} = A_i \cup \{u \cdot v : u, v \in A_i \text{ and } u \cdot v \notin A_i\}$.

Then $A = \bigcup_{i<\omega} A_i$ is a countable STS.

A is the free STS on three generators (in the sense of universal algebra).

This construction can be generalised to include the disjoint union of partial STSs in the base step.

A free construction

Given three points $\{a, b, c\}$ not forming a block, define a chain $\{A_i : i < \omega\}$ of partial STSs inductively:

- $A_0 = \{a, b, c\}$ and no products are defined
- $A_{i+1} = A_i \cup \{u \cdot v : u, v \in A_i \text{ and } u \cdot v \notin A_i\}$.

Then $A = \bigcup_{i<\omega} A_i$ is a countable STS.

A is the free STS on three generators (in the sense of universal algebra).

This construction can be generalised to include the disjoint union of partial STSs in the base step.

A free construction

Given three points $\{a, b, c\}$ not forming a block, define a chain $\{A_i : i < \omega\}$ of partial STSs inductively:

- $A_0 = \{a, b, c\}$ and no products are defined
- $A_{i+1} = A_i \cup \{u \cdot v : u, v \in A_i \text{ and } u \cdot v \notin A_i\}$.

Then $A = \bigcup_{i<\omega} A_i$ is a countable STS.

A is the free STS on three generators (in the sense of universal algebra).

This construction can be generalised to include the disjoint union of partial STSs in the base step.

A free construction

Given three points $\{a, b, c\}$ not forming a block, define a chain $\{A_i : i < \omega\}$ of partial STSs inductively:

- $A_0 = \{a, b, c\}$ and no products are defined
- $A_{i+1} = A_i \cup \{u \cdot v : u, v \in A_i \text{ and } u \cdot v \notin A_i\}$.

Then $A = \bigcup_{i<\omega} A_i$ is a countable STS.

A is the free STS on three generators (in the sense of universal algebra).

This construction can be generalised to include the disjoint union of partial STSs in the base step.

A free construction

Given three points $\{a, b, c\}$ not forming a block, define a chain $\{A_i : i < \omega\}$ of partial STSs inductively:

- $A_0 = \{a, b, c\}$ and no products are defined
- $A_{i+1} = A_i \cup \{u \cdot v : u, v \in A_i \text{ and } u \cdot v \notin A_i\}$.

Then $A = \bigcup_{i<\omega} A_i$ is a countable STS.

A is the free STS on three generators (in the sense of universal algebra).

This construction can be generalised to include the disjoint union of partial STSs in the base step.

A free construction

Given three points $\{a, b, c\}$ not forming a block, define a chain $\{A_i : i < \omega\}$ of partial STSs inductively:

- $A_0 = \{a, b, c\}$ and no products are defined
- $A_{i+1} = A_i \cup \{u \cdot v : u, v \in A_i \text{ and } u \cdot v \notin A_i\}$.

Then $A = \bigcup_{i<\omega} A_i$ is a countable STS.

A is the free STS on three generators (in the sense of universal algebra).

This construction can be generalised to include the disjoint union of partial STSs in the base step.

Smallness

Idea: we use the free construction to find 2^ω complete 3-types over $\emptyset$. This shows that T^*_{STS} does not have a countable saturated model.

Lemma

Let $\{A_i : i < \omega\}$ be a family of finite STSs such that $|A_i| \geq 3$ for at least one $i \in \omega$. Then there is a countably infinite M such that

- *M is generated by three elements*
- *every A_i embeds in M*
- *if a finite STS embeds in M, then it embeds in some A_i.*

Lemma (Doyen, 1969)

For all $n \equiv 1, 3 \pmod 6$ there is an STS of cardinality n that does not embed any STS of cardinality m for $3 < m < n$.

Smallness

Idea: we use the free construction to find 2^ω complete 3-types over $\emptyset$. This shows that T^*_{STS} does not have a countable saturated model.

Lemma

Let $\{A_i : i < \omega\}$ be a family of finite STSs such that $|A_i| \geq 3$ for at least one $i \in \omega$. Then there is a countably infinite M such that

- *M is generated by three elements*
- *every A_i embeds in M*
- *if a finite STS embeds in M, then it embeds in some A_i.*

Lemma (Doyen, 1969)

For all $n \equiv 1, 3 \pmod 6$ there is an STS of cardinality n that does not embed any STS of cardinality m for $3 < m < n$.

Smallness

Idea: we use the free construction to find 2^ω complete 3-types over $\emptyset$. This shows that T^*_{STS} does not have a countable saturated model.

Lemma

Let $\{A_i : i < \omega\}$ be a family of finite STSs such that $|A_i| \geq 3$ for at least one $i \in \omega$. Then there is a countably infinite M such that

- *M is generated by three elements*
- *every A_i embeds in M*
- *if a finite STS embeds in M, then it embeds in some A_i.*

Lemma (Doyen, 1969)

For all $n \equiv 1, 3 \pmod 6$ there is an STS of cardinality n that does not embed any STS of cardinality m for $3 < m < n$.

Smallness

Idea: we use the free construction to find 2^ω complete 3-types over $\emptyset$. This shows that T^*_{STS} does not have a countable saturated model.

Lemma

Let $\{A_i : i < \omega\}$ be a family of finite STSs such that $|A_i| \geq 3$ for at least one $i \in \omega$. Then there is a countably infinite M such that

- *M is generated by three elements*
- *every A_i embeds in M*
- *if a finite STS embeds in M, then it embeds in some A_i.*

Lemma (Doyen, 1969)

For all $n \equiv 1, 3 \pmod 6$ there is an STS of cardinality n that does not embed any STS of cardinality m for $3 < m < n$.

Smallness

Idea: we use the free construction to find 2^ω complete 3-types over $\emptyset$. This shows that T^*_{STS} does not have a countable saturated model.

Lemma

Let $\{A_i : i < \omega\}$ be a family of finite STSs such that $|A_i| \geq 3$ for at least one $i \in \omega$. Then there is a countably infinite M such that

- *M is generated by three elements*
- *every A_i embeds in M*
- *if a finite STS embeds in M, then it embeds in some A_i.*

Lemma (Doyen, 1969)

For all $n \equiv 1, 3 \pmod 6$ there is an STS of cardinality n that does not embed any STS of cardinality m for $3 < m < n$.

Smallness

Idea: we use the free construction to find 2^ω complete 3-types over $\emptyset$. This shows that T^*_{STS} does not have a countable saturated model.

Lemma

Let $\{A_i : i < \omega\}$ be a family of finite STSs such that $|A_i| \geq 3$ for at least one $i \in \omega$. Then there is a countably infinite M such that

- *M is generated by three elements*
- *every A_i embeds in M*
- *if a finite STS embeds in M, then it embeds in some A_i.*

Lemma (Doyen, 1969)

For all $n \equiv 1, 3 \pmod 6$ there is an STS of cardinality n that does not embed any STS of cardinality m for $3 < m < n$.

Theorem

T^*_{STS} does not have a countable saturated model.

Proof.

(Sketch) For $n \equiv 1, 3 \pmod 6$ let A_n be the STS of cardinality n given by Doyen's result.
Let $I = \{n \in \mathbb{N} : n \equiv 1, 3 \pmod 6\}$.
For every infinite $X \subseteq I$ let M_X be the countable STS obtained by $\{A_n : n \in I\}$ as in the previous lemma. In particular, M_X is generated by three elements.
If $X \neq Y$, M_X and M_Y are not isomorphic. Then the types of the generators of M_X and M_Y are different.
Hence there are 2^ω of these types.

□

Theorem

T^*_{STS} does not have a countable saturated model.

Proof.

(Sketch) For $n \equiv 1, 3 \pmod 6$ let A_n be the STS of cardinality n given by Doyen's result.

Let $I = \{n \in \mathbb{N} : n \equiv 1, 3 \pmod 6\}$.

For every infinite $X \subseteq I$ let M_X be the countable STS obtained by $\{A_n : n \in I\}$ as in the previous lemma. In particular, M_X is generated by three elements.

If $X \neq Y$, M_X and M_Y are not isomorphic. Then the types of the generators of M_X and M_Y are different.

Hence there are 2^ω of these types.

□

Theorem

T^*_{STS} does not have a countable saturated model.

Proof.

(Sketch) For $n \equiv 1, 3 \pmod 6$ let A_n be the STS of cardinality n given by Doyen's result.
Let $I = \{n \in \mathbb{N} : n \equiv 1, 3 \pmod 6\}$.
For every infinite $X \subseteq I$ let M_X be the countable STS obtained by $\{A_n : n \in I\}$ as in the previous lemma. In particular, M_X is generated by three elements.
If $X \neq Y$, M_X and M_Y are not isomorphic. Then the types of the generators of M_X and M_Y are different.
Hence there are 2^ω of these types. □

Theorem

T^*_{STS} does not have a countable saturated model.

Proof.

(Sketch) For $n \equiv 1, 3 \pmod 6$ let A_n be the STS of cardinality n given by Doyen's result.
Let $I = \{n \in \mathbb{N} : n \equiv 1, 3 \pmod 6\}$.
For every infinite $X \subseteq I$ let M_X be the countable STS obtained by $\{A_n : n \in I\}$ as in the previous lemma. In particular, M_X is generated by three elements.
If $X \neq Y$, M_X and M_Y are not isomorphic. Then the types of the generators of M_X and M_Y are different.
Hence there are 2^ω of these types.

□

Theorem

T^*_{STS} does not have a countable saturated model.

Proof.

(Sketch) For $n \equiv 1, 3 \pmod 6$ let A_n be the STS of cardinality n given by Doyen's result.
Let $I = \{n \in \mathbb{N} : n \equiv 1, 3 \pmod 6\}$.
For every infinite $X \subseteq I$ let M_X be the countable STS obtained by $\{A_n : n \in I\}$ as in the previous lemma. In particular, M_X is generated by three elements.
If $X \neq Y$, M_X and M_Y are not isomorphic. Then the types of the generators of M_X and M_Y are different.
Hence there are 2^ω of these types.

□

Theorem

T^*_{STS} does not have a countable saturated model.

Proof.

(Sketch) For $n \equiv 1, 3 \pmod 6$ let A_n be the STS of cardinality n given by Doyen's result.
Let $I = \{n \in \mathbb{N} : n \equiv 1, 3 \pmod 6\}$.
For every infinite $X \subseteq I$ let M_X be the countable STS obtained by $\{A_n : n \in I\}$ as in the previous lemma. In particular, M_X is generated by three elements.
If $X \neq Y$, M_X and M_Y are not isomorphic. Then the types of the generators of M_X and M_Y are different.
Hence there are 2^ω of these types.

□

Theorem

T^*_{STS} does not have a countable saturated model.

Proof.

(Sketch) For $n \equiv 1, 3 \pmod 6$ let A_n be the STS of cardinality n given by Doyen's result.
Let $I = \{n \in \mathbb{N} : n \equiv 1, 3 \pmod 6\}$.
For every infinite $X \subseteq I$ let M_X be the countable STS obtained by $\{A_n : n \in I\}$ as in the previous lemma. In particular, M_X is generated by three elements.
If $X \neq Y$, M_X and M_Y are not isomorphic. Then the types of the generators of M_X and M_Y are different.
Hence there are 2^ω of these types.

□

More about T^*_{STS}

- T^*_{STS} is TP_2 and NSOP_1
- T^*_{STS} has elimination of hyperimaginaries and weak elimination of imaginaries.

More about T^*_{STS}

- T^*_{STS} is TP_2 and NSOP_1
- T^*_{STS} has elimination of hyperimaginaries and weak elimination of imaginaries.